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BACKWARD INDUCING AND EXPONENTIAL DECAY OF CORRELATIONS FOR PARTIALLY HYPERBOLIC ATTRACTORS*

BY

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ABSTRACT

We study partially hyperbolic attractors of C^2 diffeomorphisms on a compact manifold. For a robust (non-empty interior) class of such diffeomorphisms, we construct Sinai–Ruelle–Bowen measures, for which we prove exponential decay of correlations and the central limit theorem, in the space of Hölder continuous functions. The techniques we develop (*backward inducing, redundancy elimination algorithm*) should be useful in the study of the stochastic properties of much more general non-uniformly hyperbolic systems.

1. Introduction

In this work, M will always be a compact manifold and $\mathcal{H}$ will be the space of Hölder continuous functions on M. Moreover, μ_0 is a *physical* or *SRB* (*Sinai–Ruelle–Bowen*) probability measure for f. That is, μ_0 gives the time averages

$$\mu_0 = \lim_{n\to+\infty} \frac{1}{n}\sum_{j=0}^{n-1} \delta_{f^j(z)}, \quad \delta_p = \text{ Dirac measure at } p$$

of a set $B(\mu_0)$ of points $z \in M$ with positive Lebesgue measure. This set $B(\mu_0)$ will be called the **basin** of μ_0.

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We say that (f,μ_0) has **exponential decay of correlations in $\mathcal{H}$** if there exists $\tau<1$ and for each φ, $\psi\in\mathcal{H}$ there exists $K=K(\varphi,\psi)>0$ so that

$$|C_n(\varphi,\psi)|\le K\tau^n,\quad \text{for all } n\ge 1.$$

Our main result, Theorem A below, states that *a large class of SRB measures supported on partially hyperbolic attractors of a diffeomorphism have exponential decay of correlations in $\mathcal{H}$.* We explain this in precise terms in the next subsection.

As a consequence of the proof of Theorem A we also obtain, in Theorem B, that *these systems (f,μ_0) satisfy the central limit theorem in $\mathcal{H}$.* That is, for every $\varphi\in\mathcal{H}$

$$\Phi_n=\frac{1}{\sqrt{n}}\sum_{j=0}^{n-1}(\varphi\circ f^j-\int\varphi\,d\mu_0)$$

converges in distribution to a gaussian law $N(0,\sigma)$:

$$\mu_0\left(\{x\in\mathcal{M}:\Phi_n(x)\le a\}\right)\to\int_{-\infty}^{a}\frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{t^2}{2\sigma^2}\right)dt,\quad\text{for each } a\in\mathbb{R}.$$

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1.1 STATEMENT OF MAIN RESULTS. We take $f\colon M\to M$ to be a C^2 diffeomorphism on a compact manifold M, admitting a compact invariant subset $\Lambda\subset M$ such that:

(1) There exists an open neighbourhood $\mathcal{M}$ of Λ such that

$$\text{closure} f(\mathcal{M})\subset\mathcal{M}\quad\text{and}\quad\Lambda=\bigcap_{n=0}^{\infty}(f^n(\mathcal{M})).$$

(2) Λ is *partially hyperbolic*, meaning there exists a continuous Df-invariant splitting

$$T_\Lambda M=E^{cs}\oplus E^{uu},\quad\dim(E^{uu})>0,$$

of the tangent bundle restricted to Λ with the following properties (we fix a Riemannian metric in M, once and for all):

(a) E^{uu} is uniformly expanding:

$$\|Df^{-n}|E^{uu}_{f^n(x)}\|\le C\lambda^n,\quad\text{for every } n\ge 1 \text{ and } x\in\Lambda;$$

(b) E^{cs} is *dominated* by E^{uu}:

$$\|Df^n|E_x^{cs}\| \cdot \|Df^{-n}|E_{f^n(x)}^{uu}\| \leq C\lambda^n, \quad \text{for } n \geq 1 \text{ and } x \in \Lambda,$$

where $C > 0$ and $0 < \lambda < 1$ are independent of x and n.

These conditions imply (cf. [3], [7]) that there exists a unique foliation (or lamination) $\mathcal{F}^{uu}$ of Λ which is tangent to the strong-unstable bundle E_x^{uu}, at every $x \in \Lambda$. Its leaves are C^2 submanifolds immersed in M and the attractor Λ consists of entire leaves. On the other hand, we also suppose that:

(3) There exists an invariant center-stable foliation $\mathcal{F}^{cs}$ of a neighbourhood of Λ, tangent to the central bundle E^{cs} at every point of Λ.

We reduce the neighborhood $\mathcal{M}$ if necessary, so that it is contained in the domain of $\mathcal{F}^{cs}$. Our next assumption is

(4) $f|_\Lambda$ admits a Markov partition $\mathcal{R} = \{R_0, R_1, \ldots, R_N\}$, $N \geq 1$, with a property of topological mixing: given $i, j \in \{0, 1, \ldots, N\}$, there exists $n_0 \geq 1$ such that

$$f^n(R_i) \cap R_j \neq \emptyset \quad \text{for every } n \geq n_0.$$

Here, *Markov partition* is defined in the same way as for hyperbolic systems (see [12, Chapter 10]), with $\mathcal{F}^{cs}$ in the role of stable foliation. See section 2.1 for details. We want to stress that in our setting the iterates of a Markov partition $\mathcal{R}$ need not have diameters going to zero. In other words, the set of points $z \in \Lambda$ with a given itinerary with respect to $\mathcal{R}$ is not necessarily reduced to a single point.

We are going to distinguish two types of rectangles in $\mathcal{R}$, according to the behaviour of the center-stable direction E^{cs}. Let $\lambda_s < 1$ be fixed. We call $R_i \in \mathcal{R}$ a **good rectangle** if E^{cs} is uniformly contracting,

$$\|Df|E_x^{cs}\| \leq \lambda_s < 1,$$

at every point $x \in R_i$. All the other elements of $\mathcal{R}$ are called **bad rectangles**. On a bad rectangle Df may exhibit expansion along the center-stable directions, but it should be weak:

(5) There exists some good rectangle and, for some sufficiently small $\delta_0 > 0$, we have

$$\|Df|E_x^{cs}\| \leq 1 + \delta_0 \quad \text{for any point } x \text{ in a bad rectangle.}$$

We shall see, in Proposition 2.12, that if δ_0 is taken small enough then the center-stable bundle is *mostly contracting* in the sense of [2]: all the Lyapounov exponents along E^{cs} are negative. Our precise conditions on δ_0 are stated in Proposition 2.12, and equations (3) and (4). It depends only on λ_s, the combinatorics of the Markov partition $\mathcal{R}$, and distortion bounds for the unstable direction.

It follows from [2] (see also other references below) that f admits some SRB measure μ_0 supported in Λ, and one can check that μ_0 is unique. Actually, our methods allow us to give a new proof of these facts. Above all, we prove that (f, μ_0) has exponential decay of correlations:

THEOREM A: *Under assumptions (1)–(5), the diffeomorphism f has a unique SRB measure μ_0 supported on Λ. Moreover, such SRB measure exhibits exponential decay of correlations in the space $\mathcal{H}$ of Hölder continuous functions.*

Moreover,

THEOREM B: *The system (f, μ_0) satisfies the central limit theorem in the space $\mathcal{H}$ of Hölder continuous functions.*

Let us mention that similar results were obtained recently by Dolgopyat [6], independently and through a very different approach, for another class of partially hyperbolic attractors with mostly contracting central direction.

1.2 EXAMPLES. (1) Carvalho [4] proved existence and uniqueness of SRB measures for certain partially hyperbolic attractors (in dimension ≥ 3) obtained by modifying an Anosov diffeomorphism by isotopy (along the stable direction) in a neighbourhood of some periodic saddle point p, in such a way that this periodic point goes through a Hopf (respectively, saddle-node or period-doubling) bifurcation. The diffeomorphisms one gets in this way have a non-hyperbolic attractor satisfying conditions (1), (2), (3) of the previous subsection. With a slightly more detailed construction than was needed in [4], it is possible to ensure that these diffeomorphisms also satisfy condition (4). For this, one should fix a Markov partition $\mathcal{R}$ of the original Anosov diffeomorphism, and a periodic saddle in the interior of some Markov rectangle, and then take the isotopy with support contained in that rectangle. In this way, $\mathcal{R}$ will also be a Markov partition for f, although its refinements $\bigvee_{i=-n}^{n} f^i(\mathcal{R})$ will not have diameter going to zero. Finally, if the isotopy is carried just a little beyond the Hopf bifurcation, then the diffeomorphism satisfies condition (5) too.

This procedure extends to much more general modifications of the original Anosov diffeomorphism, other than Hopf bifurcations. The main requirement is that the perturbation should preserve partial hyperbolicity with existence of central-stable foliation $\mathcal{F}^{cs}$. We can even perturb the initial system inside several

Markov rectangles, as long as hyperbolicity is kept in at least one of them (at least one good rectangle). In fact, this approach provides open classes of systems to which our theorems apply, as we prove in Proposition 1.1 and Corollary 1.2 below.

(2) Let us begin with an Anosov C^2-diffeomorphism f_0: $T^3 \mapsto T^3$ on the 3-dimensional torus, with one expanding and two contracting directions. We suppose that the norm of Df_0 along the stable subbundle and the norm of Df_0^{-1} along the unstable bundle are bounded by a constant $\lambda_0 < 1/3$. Let p be a fixed point of f_0 and R_p a Markov rectangle (of some Markov Partition of f_0) containing p in its interior. Let $\delta > 0$ be a small constant, such that the distance $d(p, \partial R_p) > 3\delta$. Denote $V_0 = B(p, \delta/2)$. Then, in the same manner as in [2], we deform f_0^{-1} inside V_0 by an isotopy obtaining a continuous family of maps $f_\nu, 0 \le \nu \le 2$ in such a way that

- (i) The continuation p_{f_ν} of the fixed point p goes through a Hopf bifurcation, and becomes a repeller for values of ν between 1 and 2 (staying all the time inside V_0). For $\nu = 1$ we have the first moment of the Hopf bifurcation, with f_1 conjugated to f_0. We suppose that the derivative $Df_1|_{E^{cs}}$ does not expand vectors. Finally, we suppose that $Df_1|_{E^{cs}}(p_{f_\nu})$ exhibits complex eigenvalues.
- (ii) In the process, there always exist a strong-unstable cone field C^{uu} (cf. [13] for definitions) and a center-stable cone field C^{cs}, defined everywhere, such that C^{cs} contains the stable direction of the initial map f_0.
- (iii) Moreover, the width of the cone fields C^{uu} and C^{cs} are bounded by a small constant $\alpha > 0$.
- (iv) There exist a constant $\sigma > 1$ and a neighbourhood $V_1 \subset V_0 \cap W^u(p)$, such that $J^c = \| \det Df_\nu^{-1}|_{E^{cs}} \| > \sigma$ outside V_1.
- (v) The map f_ν^{-1} is $\delta - C^0$ close to f_0^{-1} outside V_0 so that $\|(Df_1^{-1}|_{E^{cs}})^{-1}\| < \lambda_0 < 1/3$ outside V_0.

Note that the properties stated in conditions (i) through (v), which are valid for $f_\nu, 0 \le \nu \le 2$, are also valid for a whole C^1-neighbourhood $\mathcal{U}$ of the set of diffeomorphisms $\{f_\nu, 0 \le \nu \le 2\}$. In particular, by [7] conditions (i) through (iii) imply that any $f \in \mathcal{U}$ has an invariant central foliation, since the central cone field enables us to define a graph transform associated to it, with domain in the space of foliations tangent to C^{cs}, which is not empty, since the stable foliation of f_0 is tangent to it. On the other hand, all $f \in \mathcal{U}$ also exhibit an unstable foliation varying continuously with the diffeomorphism.

As a consequence of lemma 6.1 of [2] there is a C^1-neighbourhood $\mathcal{U}_1 \subset \mathcal{U}$ of the set $\{f_\nu, 1 < \nu \leq 2\}$ such that for all $f \in \mathcal{U}_1$, $\Lambda = T^3 \setminus W^u(p_f)$ is a partially hyperbolic attractor, which is not hyperbolic, because it is transitive and contains a (normally hyperbolic) invariant circle. A transitive set containing an invariant circle cannot be hyperbolic, since it has points with different indexes.

Now take f in some small ball $B = B(f_1, \delta'), \delta' < \delta/2$. Suppose also that δ' is sufficiently small such that every diffeomorphism in the ball $B(f_0, \delta')$ is an Anosov conjugated to f_0 and every diffeomorphism in $B(f_1, \delta') \subset \mathcal{U}$ is partially hyperbolic. We have:

PROPOSITION 1.1: *If $\delta' > 0$ is small, $B(f_1, \delta')$ is an open set of diffeomorphisms of T^3 satisfying conditions* (1)–(5).

Proof: Suppose that f_1 is a (non-hyperbolic) diffeomorphism like above. Since $B(f_1, \delta') \subset \mathcal{U}$, all that is left to verify is that $f \in B(f_1, \delta')$ satisfies (4)–(5), if δ' is sufficiently small.

Now, given $f \in B(f_1, \delta')$ let us prove that there is an Anosov g, conjugated and close to f_0 such that $f = g$ in $T^3 \setminus V_0$. Let us consider the lifts of f, f_0, g to the covering space R^3 of T^3. We denote the lifts by the same letters of their corresponding maps in the torus. Then, we can write

$$f = f_1 + h, \quad \text{where } \|h\| < \delta'.$$

If we define $g = f_0 + h$, by our choice of δ', g is an Anosov. Moreover, since $f_1 = f_0$ in $T^3 \setminus V_0$, we have $g = f_0 + h = f$ in $T^3 \setminus V_0$. Since the Markov partition changes continuously with the point, as long as we have taken δ' small, g has a Markov partition such that $T^3 \setminus V_0$ does not intersect the boundary of any rectangle of it. So, the Markov partition of g is a Markov partition for f. Furthermore, there exists a rectangle R_g corresponding to the R_p (of f_0) in which $f = g$. This means that f has at least one good rectangle. Finally, taking δ' small, we guarantee that the eventual expansion in the central direction is not so big. So, f satisfies (1)–(5). ∎

Note that the ball $B(f_1, \delta')$ is not an open subset of non-hyperbolic diffeomorphisms. However, we also obtain:

COROLLARY 1.2: *There exists an open set of non-hyperbolic diffeomorphisms $f: T^3 \to T^3$ satisfying* (1) *through* (5).

Proof: Just take the open set of diffeomorphisms $\mathcal{U}_2 = \mathcal{U}_1 \cap B(f_1, \delta')$, δ' as in the proposition above. Conditions (1)–(5) fit for every diffeomorphism in a ball $B(f_1, \delta')$.

A transitive attractor containing a circle cannot be hyperbolic, since it must contain points of different stable indexes. So, $\mathcal{U}_1$ is an open set of non-hyperbolic diffeomorphisms satisfying (1)–(5). ∎

(3) This procedure generalizes to general non-trivial hyperbolic attractors (in the place of Anosov diffeomorphisms), e.g., Smale's solenoid; see [12, chapter 4]. Let $\mathcal{R} = \{R_0, \ldots, R_N\}$ be a Markov partition for f_0 restricted to the hyperbolic attractor Λ_0. By modifying f_0 near one of the rectangles R_0 so as to turn the contracting (stable) direction possibly slightly expanding, while keeping f_0 unchanged in $\Lambda_0 \setminus R_0$, we can obtain a partially hyperbolic attractor satisfying conditions (1) through (5).

(4) Recently, Bonatti-Viana [2] extended the results of [4] to general partially hyperbolic attractors with mostly contracting central direction: there always exist SRB measures supported in the attractor, and they are finitely many. The last section of their paper also contains several robust examples, not necessarily close to hyperbolic systems, to which our Theorems A and B apply.

1.3 Structure of the proof. Let us talk about the ideas in the proof of our results.

To deal with the non-hyperbolic behaviour of our maps (the fact that the bundle E^{cs} may fail to be contracting), we begin by constructing a new dynamical system F *induced* from the original f. That is, F is given locally by an (variable) iterate of f. This is now a standard tool in ergodic theory. However, our method is novel in that inducing (in fact, a tower construction) is carried out *backwards*. This is related to the fact that it is along the center-stable direction that hyperbolicity breaks down. More precisely, given a point $x \in \Lambda$, we analyse the negative orbit $f^{-n}(x)$ until finding some $n(x) \geq 1$ so that f^j is (uniformly) hyperbolic at $f^{-n(x)}(x)$, for every $1 \leq j \leq n(x)$. Then we construct a *tower space*

$$\mathcal{T} = \{(f^{-i}(x), -i) : 0 \leq i < n(x)\}$$

and we lift f^{-1} to a map G on $\mathcal{T}$: $\mathrm{proj} \circ G = f^{-1} \circ \mathrm{proj}$, where $\mathrm{proj} : \mathcal{T} \to \Lambda$ is the canonical projection $\mathrm{proj}(x, -i) = x$. A key point is that G is uniformly hyperbolic with respect to some metric in the tower $\mathcal{T}$. The strategy is to deduce properties of f from properties of G by means of the projection proj.

However, proceeding from this, we are faced with a serious problem: since the map G is not injective, in general, f cannot be lifted to a map on the tower (although it lifts to a multivalued relation). In order to completely bypass this difficulty, we introduced a *redundancy elimination algorithm*. We construct $\mathcal{T}$

as a union of local center-stable leaves (intersected with the attractor). In brief terms, the algorithm picks exactly one (convenient) copy of each center-stable leaf in the tower, and removes all the others. The "trimmed" tower obtained after the algorithm is applied is isomorphic to a full measure subset of Λ, and f lifts to a map G^{-1} in it, which is an embedding.

The negative iterates of the remaining center-stable leaves which are contained in the 0-th floor of the trimmed tower exhausts (using f^{-1} iterates) Λ, up to a zero measure (for the conditional Lebesgue measure of any unstable leaf) subset of Λ. The advantage of this algorithm is that we decompose almost every orbit into minimal possible segments so that the left endpoint is a hyperbolic time (cf. Definition 2.4) for the right endpoint. Both endpoints are in the 0-th floor (that is, the subset of points $(y, 0) \in \mathcal{T}$) of the tower. Moreover, the first return map (obtained using positive iterates of f) of this floor has Markovian properties.

Sometimes we identify sets in the tower with their isomorphic images (under proj) in Λ. The 0-th floor of the trimmed tower and its isomorphic image are both denoted by $\mathcal{A}$.

Due to the way we construct our tower, this set $\mathcal{A}$ is hyperbolic in a strong sense: there is $\varsigma < 1$ such that, given any positive iterate i, and $x \in \mathcal{A}$, we have $\|Df^i|E^{cs}(x)\| \leq \varsigma^i$. Once we reach this point, the main results of our work can be derived along fairly well-known lines, from the properties of set $\mathcal{A}$.

In the third section of our paper, we study the return map F (or *root map*) of $\mathcal{A}$, centering our attention on examples such that the Markov rectangles are extensible to the ambient (this is the case of examples (1), (2) and (3) above). In such cases, the SRB measure can be obtained as an exponentially fast limit of iterates of Lebesgue measure in the ambient space M restricted to the positive Lebesgue measure subset $\mathcal{A}$.

For that purpose, we adapt techniques in [8], to construct a measure ($\mu_{\mathcal{A}}$) invariant for this return map. Indeed, $\mu_{\mathcal{A}}$ is, up to a normalization factor, the SRB measure μ_0 of f restricted to $\mathcal{A}$. We construct the SRB measure μ_0 for f, just pushing $\mu_{\mathcal{A}}$ forward to the whole Λ.

In section 4, we show a more abstract way (independent from that in section 3) to construct the SRB measure and prove that it is Kolmogorov. Finally, we deduce that (f, μ_0) has exponential decay of correlations in the space of Hölder continuous functions and satisfies the central limit theorem.

Besides the results we prove here, we expect these methods of backward inducing and redundancy elimination algorithm to be useful in much more generality, especially to study systems whose prospective stable direction fails to be contracting.

2. Construction of the set $\mathcal{A}$

2.1 SOME NOTATIONS AND DEFINITIONS. We begin by making precise what we mean by Markov partition of a diffeomorphism f on a partially hyperbolic attractor Λ. The standing assumption is that f has a center-stable foliation $\mathcal{F}^{cs}$ defined on a neighbourhood $\mathcal{M}$ of Λ, besides the strong-unstable foliation.

Definition 2.1 (Proper rectangle): A **proper rectangle** R of Λ is a closed subset of it such that:

1. R is the closure of its interior (in the relative topology of Λ).
2. Given $x, y \in R$, there is exactly one point $z = [x, y]$ in R which is the intersection of the local center-stable manifold through x and the local unstable manifold through y. Moreover, the bracket map $[.,.]$ is continuous in R.

Definition 2.2 (Markov partition): A collection $\mathcal{R} = \{R_0, R_1, \ldots, R_N\}$, $N \geq 1$ of proper rectangles is a **Markov partition** for f if:

1. $\bigcup_{i=0}^{N} R_i = \Lambda$ and the interiors of the R_i's are two-by-two disjoint.
2. If the intersection $\Gamma \cap \Lambda$ of a local unstable manifold with the set Λ is contained in a rectangle R_i, and $f^{-1}(\Gamma) \cap R_j \neq \emptyset$, for some $j \in \{0, \ldots, N\}$, then $f^{-1}(\Gamma) \subset R_j$. A similar property is satisfied by any local central manifold γ such that $\gamma \cap \Lambda$ is contained in R_i: if $f(\gamma \cap \Lambda) \cap R_j \neq \emptyset$, $j \in \{0, \ldots, N\}$, then $f(\gamma) \subset R_j$.

Above, by local unstable manifold and local center-stable manifold we just mean small neighbourhoods of the point inside the corresponding unstable and center-stable leaves. From now on, these expressions will have a slightly more precise meaning: unless we say otherwise, we will use **local unstable manifold** or just **unstable manifold** to mean the intersection of the local unstable manifold of a point $x \in \Lambda$ with the rectangle of the Markov partition that contains Λ. A similar convention will apply to the expression **local center-stable manifold** (or **local central manifold**). In other words, the expression **local center-stable manifold** will designate points in Λ with the same future ($n \geq 0$) itinerary with respect to the Markov partition $\mathcal{R}$ of Λ. Analogously, we will use local unstable manifold for points with the same past ($n \leq 0$) itinerary with respect to $\mathcal{R}$.

Local central manifolds will be represented by the (lower case) greek letter γ. If $f^i(\gamma) \subset \gamma'$, for some $i \in \mathbb{N}$, we say γ' is an **ancestor** (for f^{-1}) of γ.

Sometimes we are going to deal with functions which are constant in local central manifolds. If h is a such function we will denote by $h(\gamma)$ its value in γ.

For each rectangle R_i of the Markov partition $\mathcal{R}$, we will call

$$\kappa_i = \kappa(R_i) = \sup_{x \in R_i} (\|Df|E^{cs}(x)\|).$$

$R(x)$ will be the rectangle of the partition that contains x, and if $x \in R_i$, we set $\kappa(x) = \kappa_i$.

Given $n \in \mathbb{N}$, we will call an n-**cylinder** a set C of points such that each $f^i(C)$ is entirely contained in a rectangle, for $i = 0, \ldots, n$. When n is assumed, we drop it, and we just write **cylinder** instead of n-**cylinder**.

Fixed a rectangle R (and $n \in \mathbb{N}$); we will refer to the set of cylinders C such that $f^n(C) \subset R$ as the **cylinders finishing at** R.

We will use the term **unstable leaf** to mean any restriction of a local unstable manifold, or any countable union of local unstable manifolds. We will use the term **central leaf** to mean any finite union of local central manifolds.

As we have seen in the introduction of this work, we are going to construct a kind of tower structure from the original map f. Let us say what we mean by **tower** in this work.

Definition 2.3 (Tower space)*:* A **tower space** $\mathcal{T}$ or simply a **tower** $\mathcal{T}$ is a countable disjoint union $\bigcup_l (S_l, l)$ of copies of sets S_l in Λ, where each S_l is a union of local central manifolds, and l is a non-positive integer. Fix a non-positive integer l'; the set $(S_{l'}, l')$ is called the l'**-th floor** of $\mathcal{T}$. The **0-th floor** is also called **ground floor**.

2.2 Backward inducing. Now we can begin the construction of our tower. This will be done by means of **backward inducing**, as explained in the introduction.

In the 0-th floor of the tower, we include a copy $(\gamma, 0)$ of each $\gamma \in \mathcal{F}^{cs}_{loc}$. So, the 0-th floor is, in fact, a copy of Λ. In the sequel, we will often identify γ and $(\gamma, 0)$. We associate to each $(\gamma, 0)$ the constant function $\omega_0((\gamma, 0)) = 1$. Now, suppose the l-th floor is constructed, and that we have associated to it a function $\omega_l((\gamma, l))$, constant in each central leaf in it. Given (γ, l), denote $\gamma_i, i = 1, \ldots, r$ the local central manifolds that participate in the pre-image of γ by f (that is, these are the local central leaves whose image under f is contained in γ). Let $\varsigma > \lambda_s$ be any constant less than one. For each γ_i we calculate the expression

$$E_i = \omega_l((\gamma, l)) \cdot \kappa(\gamma_i) \cdot \varsigma^{-1}.$$

If E_i is less than or equal to 1, we do nothing; otherwise, we include a copy $(\gamma_i, l-1)$ of γ_i in the $(l-1)$-st floor. In other words, the set S_{l-1} (see tower definition, in the last subsection) is precisely the union of all local central manifolds γ_i for which $E_i > 1$. For those γ_i we define

$$\omega_{l-1}(\gamma_i, l-1) = E_i.$$

This procedure defines our tower space.

In a natural manner, we lift f^{-1} to a tower map G on the tower, that is, G is such that

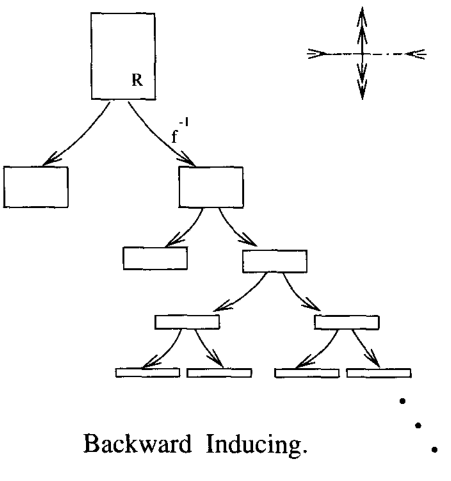
$$\mathrm{proj}(G(x,l)) = f^{-1}(\mathrm{proj}(x,l))$$

where $\mathrm{proj}(x,l) = x$.

In fact, G is defined as

$$G(x,l) = \begin{cases} (f^{-1}(x), l-1), & \text{if } \omega_l(x,l)\cdot\kappa(f^{-1}(x))\cdot\varsigma^{-1} > 1, \\ (f^{-1}(x), 0), & \text{otherwise.} \end{cases}$$

However, the inverse G^{-1} is not well defined as a function, because G is not injective.



Backward Inducing.

In the next definition, we adapt to our context a notion introduced in [1]:

Definition 2.4 (Hyperbolic time): Given $z \in \Lambda$, $-p$ is a **hyperbolic time** for z if there exists $p \in \mathbb{N}$ such that $f^{-p}(z) = y$ and, for $i = 1, \ldots, p$,

$$\varsigma^{-i}\cdot\prod_{j=0}^{i-1}\sup_{x\in R(f^j(y))}\|Df|E^{cs}\| = \varsigma^{-i}\cdot\prod_{j=0}^{i-1}\kappa(f^j(y)) \le 1 \tag{1}$$

holds. This means $Df^i|E^{cs}(y)$ is contracting for $i = 1, \ldots, p$ by a factor of contraction at least of ς^i.

We say $-p$ is the **first hyperbolic time** for z, if none of $-p+1, \ldots, -1$ is hyperbolic time for z.

Sometimes, by a slight abuse of language, we say $y = f^{-p}(z)$ is the first hyperbolic time for z to denote that $-p$ is the **first hyperbolic time** for z.

Remark 2.5: Note that replacing y by any $y' \in \gamma(y)$ does not affect the value of (1). Hence y' is (first) hyperbolic time for $f^p(y')$, if and only if y is (first) hyperbolic time for $z = f^p(y)$.

Definition 2.6 (A sequence of γ)*:* We say that $\{(\gamma^0, 0), (\gamma^{-1}, -1), \ldots, (\gamma^{-k}, -k), (\gamma^{-k-1}, 0)\}$ forms **a sequence of** $(\gamma, 0) = (\gamma^0, 0)$ if $(\gamma^{-i}, -i) \subset G(\gamma^{-i+1}, -i+1)$, for $i = 1, \ldots, k$ and $(\gamma^{-k-1}, 0) \subset G(\gamma^{-k}, -k)$.

Definition 2.7 (The tree of γ)*:* The **tree** of γ, denoted by $T(\gamma)$, is the union of all sequences of γ. In this case, we will call $\gamma \simeq (\gamma, 0)$ the **root** of the tree. Each element $(\gamma^k, k), k \neq 0$ of the tree will be called a **branch** of the tree. The last term of each sequence of γ will be called a **flower** of the tree. Sometimes, we will refer to a $(\tilde{\gamma}, i)$ in $T(\gamma)$ as a **leaf** of $T(\gamma)$, where $(\tilde{\gamma}, i)$ can be a branch, a flower or the root.

Remark 2.8: It is a trivial but important remark that if γ', γ'' belongs to the same rectangle of the Markov partition, then their trees are combinatorially isomorphic (their trees have the same combinatorial feature).

We are able to prove the following important result:

PROPOSITION 2.9: *The flowers of the tree of $(\gamma, 0)$ are the first hyperbolic time for $(\gamma, 0)$. That is, if $\{(\gamma^0, 0), (\gamma^{-1}, -1), \ldots, (\gamma^k, k), (\gamma^{k-1}, 0)\}$ is a sequence of the tree of γ, then the points of γ^{k-1} are the first hyperbolic time for their respective images in $f^{-k+1}(\gamma^{k-1}) \subset \gamma$.*

Proof: Consider the sequence of γ expressed above. By the construction of our tower, we know that

$$\varsigma^l \cdot \prod_{j=0}^{-l-1} \kappa(f^j(\gamma^l)) = \omega_l((\gamma^l, l)) > 1.$$

This means that the points of γ^l are not hyperbolic times for their respective f^{-l}-images in γ, for $k-1 < l < 0$. So, we conclude that going backwards up to $l = k$ we do not have hyperbolic time.

Therefore, the only thing left to prove is that we have hyperbolic time (which then will be the first) for $l = k - 1$.

We know that if $k - 1 < r \le -1$,

$$1 \ge \varsigma^{k+1} \cdot \prod_{j=0}^{-k} \kappa(f^j(\gamma^{k-1})) = \varsigma^r \cdot \prod_{j=0}^{-r-1} \kappa(f^j(\gamma^{k-1})) \cdot \omega_r((\gamma^r, r).$$

Since $\omega_r((\gamma^r, r)) > 1$, and the product above is ≤ 1, the factor $\varsigma^r \cdot \prod_{j=0}^{-r-1} \kappa(f^j(\gamma^{k-1}))$ has to be ≤ 1. So, we have

$$\varsigma^{-i} \cdot \prod_{j=0}^{i-1} \kappa(f^j(y)) \le 1, \quad \forall y \in \gamma^{k-1}, \quad 1 \le i \le -k+1. \qquad \blacksquare$$

Our next step is proving that, given any local unstable manifold Γ, supplied with its respective Lebesgue measure m_u, the measure m_u of (the projection of) the n-th floor of the tower, intersected with Γ, decreases exponentially fast with n.

For this purpose, we will need the following result of V. Pliss, which assures that the hyperbolic times are quite common. See, e.g., [9] for a proof.

LEMMA 2.10 (Pliss lemma): *Given $\lambda > 0$, $\epsilon > 0$, $H > 0$, $\exists N_0 = N_0(\lambda, \epsilon, H)$, $\delta = \delta(\lambda, \epsilon, H) > 0$ such that, if $a_1, \ldots, a_{N_1}$ are real numbers, $N_1 \ge N_0$ with*

$$\sum_{n=1}^{N_1} a_n \le N_1 \cdot \lambda \quad \text{and} \quad |a_n| \le H, \quad \textit{for } n = 1, \ldots, N_1,$$

then there are $1 \le n_1 \le \ldots \le n_t \le N_1$ such that

$$\sum_{i=n_j+1}^{n} a_i \le (n - n_j)(\lambda + \epsilon), \quad \forall j = 1, \ldots, t_1 \quad \text{and} \quad n_j < n \le N_1.$$

Furthermore, t_1 satisfies $t_1/N_1 \ge \delta$.

Let us explain the way we will use the Pliss lemma. If we fix any ς' such that $\lambda_s < \varsigma' < \varsigma$, then there exists N_0 (depending on ς, ς' and on bounds for $\|Df|E^{cs}\|$) such that for any point x satisfying

$$\|Df^{N_1}|E^{cs}(x)\| < (\varsigma')^{N_1}, N_1 > N_0$$

we will have at least t_1 hyperbolic times between x and $f^{N_1}(x)$, t_1 being a fixed fraction of N_1.

We will also need the following lemma:

LEMMA 2.11 (Bounded distortion in unstable directions): *Given $L_1 > 0$, there exists $\tilde{K} > 0$ such that, given any C^2 disk Γ with tangent bundle inside an unstable cone field and with small curvature, and given any $n \geq 1$ such that* $\operatorname{diam}(f^n(\Gamma)) < L_1$, *we have*

$$\frac{1}{\tilde{K}} \leq \frac{|\det Df^n|_{T_x\Gamma}(x)|}{|\det Df^n|_{T_y\Gamma}(y)|} \leq \tilde{K},$$

for every x, $y \in \Gamma$.

Proof: Let us write $J^n_\Gamma(z) = |\det Df^n|_{T_z\Gamma}|$. Due to lemma 3.1 of [2], the positive iterates $f^i(\Gamma)$ have bounded curvature, which implies that $J_{f^i(\Gamma)}$ is c'-Lipschitz continuous for some uniform constant $c' > 0$. On the other hand, f is uniformly expanding along any direction contained in the unstable cone field, so we have

$$d(f^i(x), f^i(y)) \leq \lambda_u^{n-i} d(f^n(x), f^n(y)) \leq \lambda_u^{n-i} L_1, \quad i = 0, \ldots, n.$$

Therefore, we obtain

$$\begin{aligned}\left|\log \frac{J^n_\Gamma(x)}{J^n_\Gamma(y)}\right| &\leq \sum_{i=0}^{n-1} |\log J_{f^i(\Gamma)}(f^i(x)) - \log J_{f^i(\Gamma)}(f^i(y))| \\ &\leq \sum_{i=0}^{n-1} c' d(f^i(x), f^i(y)) \leq \sum_{i=0}^{n-1} c'(\lambda_u^{n-i} L_1).\end{aligned}$$

Hence, just take $\tilde{K} = \exp(c' \cdot L_1 \cdot \sum_{i=0}^{\infty} \lambda_u^i)$. ∎

PROPOSITION 2.12: *Given any local unstable manifold Γ, the intersection of the n-th floor S_n of the tower with Γ has exponentially small m_u measure, as long as we fix δ_0 in condition (5) (see introduction) sufficiently close to zero. That is, as long as we fix constant δ_0 sufficiently close to zero, there are constants $c > 0$ and $0 < u < 1$ which do not depend on Γ, such that*

$$m_u(S_n \cap \Gamma) < c \cdot u^n, \quad \forall n \in \mathbb{N}.$$

In particular, we have $\sum_{n=0}^{\infty} m_u(\Gamma \cap S_n) < +\infty$.

Proof: Let us call ν the borelian measure on Λ given by $\nu(B) = m_u(\Gamma \cap B)$, where B is any borelian set.

Let $C_1, \ldots, C_v$ be all the n-**cylinders**. Given ς' such that $\lambda_s < \varsigma' < \varsigma$ we define a **good** cylinder as one which satisfies

$$(\varsigma')^{-n} \cdot \prod_{j=0}^{n-1} \kappa(f^j(C_q)) \leq 1, \quad q \in \{1, \ldots, v\}. \tag{2}$$

If a cylinder is not good, we say it is **bad**.

Notice that there exists $N_0 \in \mathbb{N}$ such that the $(-n)$-th floor of the tower is contained in the union of the bad cylinders, if $n > N_0$. This happens because of the Pliss lemma. If γ is in a good cylinder, by the Pliss lemma, an **ancestor** of it has already become a hyperbolic time (with ς) for $f^n(\gamma)$, therefore it returned to the zero level of the tower.

Of course the sum $\sum_{i=0}^{N_0}$ of the ν-measure of the floors from the ground floor to the $-N_0$-th floor ($N_0 \in \mathbb{N}$ fixed in the paragraph above) is finite. So, the only thing we have to do is to prove that the ν-measure of the union of bad n-cylinders is exponentially small with n, $n > N_0$.

Let us consider $C(t_1, \ldots, t_r), 0 \leq t_1 < \cdots < t_r \leq n$ the union of all n-cylinders C whose images f^{t_i}, $i \in \{1, \ldots, r\}$ are contained in bad rectangles. We have:

CLAIM: *The ν-measure of $C(t_1, \ldots, t_r)$ is bounded by const $\cdot u_2^r$, for some $0 < u_2 < 1$.*

Let us prove by recurrence the above claim. Since $C(t_1, \ldots, t_r)$ is contained in $C(t_1, \ldots, t_{r-1})$, all we have to do is to verify that $m(C(t_1, \ldots, t_r)) \leq u_2 \cdot m(C(t_1, \ldots, t_{r-1}))$, for some constant $0 < u_2 < 1$. Since the Markov partition $\mathcal{R}$ is topologically mixing (and the fact that, if some positive iterate of f is exponentially mixing, so also is f) there is no loss of generality in supposing that the first positive iterate of each rectangle in $\mathcal{R}$ crosses some good rectangle in $\mathcal{R}$. Then, any disk with bounded curvature and with its tangent bundle inside the unstable cone field, crossing some rectangle, will have at most a uniform fraction $0 < u_0 < 1$ of (the conditional volume of) its image crossing bad rectangles.

On the other hand, $C(t_1, \ldots, t_{r-1})$ consists in a union of $t_r - 1$ cylinders. Let Γ_1 be the intersection of Γ with one of these $t_r - 1$ cylinders, say C. We will show that the fraction of Γ_1 occupied by $C(t_1, \ldots, t_r)$ is uniformly less than 1. In fact, for Γ_1 as above, we have (m_u is the conditional volume of disks whose tangent bundle is inside the unstable cone field)

$$m_u(f^{t_r}(\Gamma_1) \cap f^{t_r}(C(t_1, \ldots, t_r))) \leq u_0 \cdot m_u(f^{t_r}(\Gamma_1)).$$

By the bounded distortion lemma above, we obtain

$$m_u(\Gamma_1 \cap C(t_1, \ldots, t_r)) \leq u_1 \cdot m_u(\Gamma_1).$$

Finally, integrating over all small subsets Γ_1 of Γ, crossing $t_r - 1$ cylinders of $C(t_1, \ldots, t_{r-1})$ we obtain

$$\nu(C(t_1, \ldots, t_r)) \leq u_2 \cdot \nu(C(t_1, \ldots, t_{r-1})), \quad \text{with } 0 < u_2 < 1,$$

which finishes the proof of the claim.

Now, we are able to prove that the ν-measure of the union of the bad n-cylinders goes exponentially fast to 0 as n goes to infinity.

We just group the n-cylinders into sets $C(t_1,\dots,t_r), t_r \leq n$, as above. The bad sets, which group the bad cylinders, consist of points that visit the bad rectangles for a number of times greater than or equal to $r > \alpha_0 \cdot n$, where α_0 is a fixed fraction of n such that

$$\varsigma' \leq (1+\delta_0)^{\alpha_0} \cdot \lambda_s^{1-\alpha_0} < 1. \tag{3}$$

Each of them has exponentially small Lebesgue measure (bounded by const $\cdot u_2^r$), and the number of them is bounded by

$$\sum_{r>\alpha_0 \cdot n}^{n} \binom{n}{r}.$$

By Stirling's formula, we know that

$$\binom{n}{r} = \frac{n!}{(n-r)!r!} \leq \widehat{A} \cdot \frac{n^n}{(n-r)^{(n-r)} \cdot r^r},$$

for some universal constant $\widehat{A}$. Then, we have

$$\frac{n^n}{(n-r)^{n-r} \cdot r^r} = \left(\frac{n}{r}\right)^r \left(\frac{n}{n-r}\right)^{n-r} = \left[\left(1+\frac{n-r}{r}\right)\left(1+\frac{r}{n-r}\right)^{\frac{n-r}{r}}\right]^r.$$

Since $r > \alpha_0 \cdot n$, if we take α_0 near 1, the term in brackets above can be made (uniformly) as close to 1 as we want.

This implies that the ν-measure of the bad cylinders is bounded by

$$const \cdot \sum_{r>\alpha_0 \cdot n} \binom{n}{r} u_2^r < const \cdot u_3^{\alpha_0 \cdot n}, \tag{4}$$

for some $u_2 < u_3 < 1$ as long as we take α_0 close to 1. (The term **const** does not represent necessarily the same positive constant in both sides above.) ∎

COROLLARY 2.13: *Let $B = \{\gamma$, there is an infinite number of subscripts i such that (γ, i) is in the tower$\}$. Then for any (global) unstable leaf Γ, $m_u(B \cap \Gamma) = 0$.*

Proof: Indeed, this is a consequence of the Borel–Cantelli Lemma. Given any negative integer n, $B \subset \bigcup_{j=n}^{-\infty} S_j$, where (S_j, j) is the j-th floor of our tower. Given any local unstable leaf $\Gamma_1 \subset \Gamma$, we have from the proof above that such a union has exponentially small ν-measure (here, ν is the restriction of m_u with Γ_1). As long as we let $n \to -\infty$, we obtain $\nu(B) = 0$. Since this is valid for any $\Gamma_1 \subset \Gamma$, this implies that $m_u(B) = 0$. ∎

Remark 2.14: We note that, if a local stable leaf γ is contained in B, $\gamma' \supset f^j(\gamma)$ also is in B, for $j \in \mathbb{N}$. So we have

$$\mathcal{D} = \bigcup_{j \in -\mathbb{N}} f^j(B) = \bigcup_{j \in \mathbb{Z}} f^j(B).$$

Such a set $\mathcal{D}$ is f and an f^{-1}-invariant set, which is the union of local central manifolds and still has zero (conditional) Lebesgue measure, since f is a C^1-diffeomorphism.

For the sequel of our arguments, we must discard the set $\mathcal{D}$. Therefore, we will consider the tower built for $\Lambda \setminus \mathcal{D}$.

The trees constructed for such a set have an interesting property:

Proposition 2.15: *Let γ' be a central manifold in $\Lambda \setminus \mathcal{D}$. If $(\gamma, i) \in T(\gamma')$, then there is no other copy (γ, j) of γ in $T(\gamma')$, $j \neq i$.*

Proof: First, suppose both i and j to be non-zero numbers. There's no loss of generality if we suppose $j < i$. Then, we will have a copy $(\gamma', j - i)$ of γ' in $T(\gamma')$. In this case, we will have $(\gamma', n \cdot (j - i))$ in $T(\gamma')$, $\forall n \in \mathbb{N}$. Then, $\gamma' \in \mathcal{D}$.

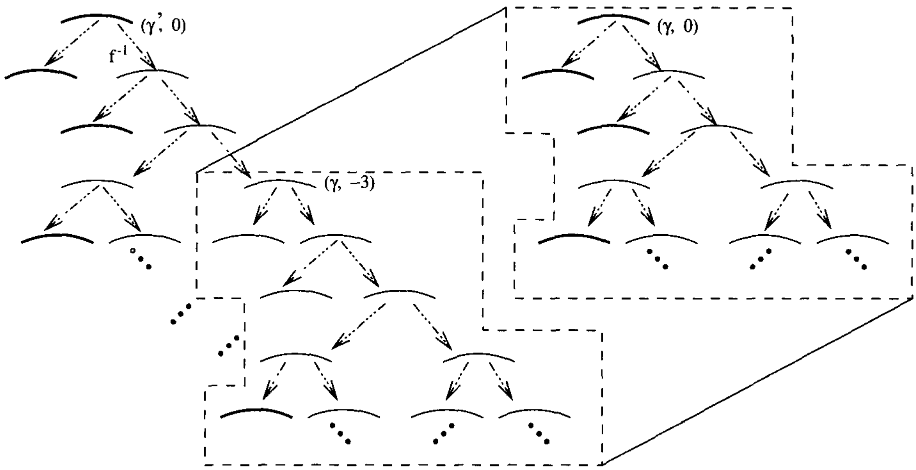
Now suppose $i = 0$, $j < 0$. If $f(\gamma) \subset \gamma'$, then $j < -1$, and again we have copy (γ', i), $i < 0$ of γ', and $\gamma' \in \mathcal{D}$. Otherwise, $f(\gamma) \subset \gamma^l$, which implies $(\gamma^l, l) \in T(\gamma')$, with $l < 0$ and $l \neq j - 1$. Then we have two different copies with non-zero subscripts of γ^l. And we are again in the first case treated above. ∎

2.3 Redundancy elimination algorithm. As we have seen, f does not lift to a map on the tower. In this subsection we describe an algorithm that permits us to discard all copies but one of each manifold (γ, i) in the tower. In fact, the union of the remaining copies will be isomorphic to $\Lambda \setminus \mathcal{D}$, a full Lebesgue measure subset of Λ. We close this subsection by analysing the action of f on the reduced tower.

First, we need some definitions.

Definition 2.16 (Tree inclusion)*:* We say the tree of γ is **contained** in the tree of γ' if there is a copy $(\gamma, i), i \neq 0$ of γ in the tree of γ'. In this case, we write $T(\gamma) \prec T(\gamma')$.

This terminology is justified by the following proposition:



Tree inclusion and redundance elimination algorithm. Each curved segment represents a local stable manifold in the tower, and we draw two trees, where the tree of gamma is contained in the tree of gamma'. Roots and flowers appear as thicker segments.

Proposition 2.17: *If $T(\gamma) \prec T(\gamma')$, then for every leaf $(\tilde{\gamma}, i)$ in $T(\gamma)$ there will be a copy $(\tilde{\gamma}, j)$ in $T(\gamma')$, with $i \geq j$ (recall that i, j are non-positive). If $i = j$, then both are zero.*

Proof: Suppose $i \neq 0$. Let $(\tilde{\gamma}, i) \in T(\gamma)$ be as in the statement of the proposition. If $T(\gamma) \prec T(\gamma')$, there exists $k \neq 0$ such that $(\gamma, k) \in T(\gamma')$. Then, $f^{-i}(\tilde{\gamma}) \subset \gamma$ implies that $f^{-(i+k)}(\tilde{\gamma}) \subset \gamma'$. Since $f^{-(i+k)}(\tilde{\gamma}) \subset \gamma'$ does not have hyperbolic time up to k, it cannot have hyperbolic time up to $k + i$, because, if it had one, such a hyperbolic time would correspond to a hyperbolic time for $f^{-i}(\tilde{\gamma}) \subset (\gamma, 0)$ before i, and $f^{-i}(\gamma)$ does not reach its first hyperbolic time up to i.

Therefore, if we put $j = k + i$, we will have $(\tilde{\gamma}, j) \in T(\gamma')$, by the rules of the tree construction. In this case, we see that $i \neq j$.

If $i = 0$, we can apply our last analysis for $(\gamma^l, l) \supset f(\tilde{\gamma}, 0)$, $(\gamma^l, l) \in T(\gamma)$, $l < 0$. (We can suppose $l < 0$, because, if $l = 0$, $\gamma^l = \gamma$, and the result is trivial.)

So, either $(\tilde{\gamma}, 0)$ belongs to $T(\gamma')$ (so $j = i = 0$) or $(\tilde{\gamma}, l - 1)$ belongs to $T(\gamma')$ (in this case, $j = l - 1 \neq 0$). ∎

Now, we can explain our method of redundancy elimination. Given a tree $T(\gamma)$ it can be *contained* in, at most, a finite number of other trees. This is because we discard the invariant set $\mathcal{D}$, which contains leaves with an infinite number of copies in the initial tower. So, every local central manifold in the tower has just a finite number of copies. If $T(\gamma)$ was **contained** in an infinite number of other trees, γ would have an infinite number of copies, so it would belong to $\mathcal{D}$ and so would also do the manifolds whose copies lie in its tree.

Then, given γ, γ' such that $T(\gamma) \prec T(\gamma')$, we just erase $T(\gamma)$ from the tower, except the flowers that are common to both trees. In other words, we stay only with $T(\gamma')$. We will not lose any information by proceeding so, because, as we have seen in the last proposition, there is a copy of $T(\gamma)$ in $T(\gamma')$, just with decreased subscript (recall that the subscripts are negative) in relation to the original $T(\gamma)$. Since $T(\gamma)$ is *contained* in just a finite number of trees, this process finishes in some steps.

Of course, by our algorithm, we always keep at least a copy of each local central manifold γ in the tower, as we just eliminate some copy if we keep at least another as back-up.

In fact, there will be exactly one copy of each $\gamma \in \Lambda \setminus \mathcal{D}$, in the final of the process, as we show in the next proposition.

Let us call **maximal** the trees that were not eliminated by our process. The following result justifies this terminology:

PROPOSITION 2.18: *Let $T(\gamma')$ be a maximal tree. If $(\tilde{\gamma}, i)$ is in $T(\gamma')$, it cannot have (a different) copy in any other maximal tree.*

Proof: Suppose it has a copy $(\tilde{\gamma}, j) \in T(\gamma)$, $T(\gamma)$ maximal too, and $i \neq j$. First, suppose $i < 0$. If $j < i$, $(\gamma, j - i)$ would belong to $T(\gamma')$, so $T(\gamma) \prec T(\gamma') \Rightarrow T(\gamma)$ is not maximal. The same is valid if $j > i$, $j \neq 0$, exchanging the roles played by γ and γ'. If $i = 0$, then $j < 0$ (in order to have $(\tilde{\gamma}, j) \neq (\tilde{\gamma}, 0)$). Therefore, $f^{-j}(\tilde{\gamma}) \subset \gamma'$ has not reached its first hyperbolic time until j. So $(\tilde{\gamma}, 0)$ is first hyperbolic time for $f^{-k}(\tilde{\gamma}) \subset \gamma$, $k > j$. Hence, $(\gamma, j - k) \subset T(\gamma')$, and again we have $T(\gamma) \prec T(\gamma')$, which contradicts $T(\gamma)$'s maximality. ∎

Finally, the following proposition assures that our maximal trees glue nicely:

PROPOSITION 2.19: *If $(\gamma, 0)$ is a flower in a maximal tree $T(\gamma')$, it is also the root of another maximal tree.*

Proof: Suppose the contrary. So, there will be a copy (γ, i), $i \neq 0$ of γ in another tree, for instance, $T(\gamma'')$. If $f^{-j}(\gamma) \subset \gamma'$ $(j < 0)$, we must have $j < i$, otherwise $T(\gamma') \prec T(\gamma'')$, which cannot occur, since $T(\gamma')$ is maximal.

Therefore, $(\gamma'', j - i)$ belongs to $T(\gamma')$, hence $T(\gamma'') \prec T(\gamma')$. By our Proposition 2.15, each leaf in $T(\gamma'')$ has a subscript greater than or equal to the subscript of its copy in $T(\gamma')$ (recall that the subscripts are nonpositive integers). This implies that $i \geq 0$, which is a contradiction. ∎

Let $\mathcal{S} \simeq \Lambda \setminus \mathcal{D}$ be the set of tower surviving leaves (that rested after we apply our elimination algorithm) and let $\mathcal{A}$ be the set

$$\mathcal{A} := \{(\gamma, 0) \in \mathcal{S}\}.$$

We will keep the identification $\gamma \simeq (\gamma, 0)$.

Now, we can lift f to $\mathcal{S}$ (we will use the same symbol f for the lift):

$$f\colon \mathcal{S} \hookleftarrow$$
$$f(x,l) = \begin{cases} (f(x), l+1), & \text{if } l < 0, \\ (f(x), j) & \text{otherwise}, \end{cases}$$

where j is the only one such that $(f(x), j) \in \mathcal{S}$.

Except for the fact that we have discarded $\mathcal{D}$, we are back to a diffeomorphism on Λ. (Indeed, in the next sections we will always write Λ, instead of $\mathcal{S}$.) However, the trimmed tower structure permits us to decompose almost every orbit into minimal segments, whose endpoints are the return times to the 0-th floor $\mathcal{A}$. We are led to consider the **root map** induced by f, defining F: $\mathcal{A} \hookleftarrow$ as the first return map of $\mathcal{A}$. This is the same map that takes flowers in $\mathcal{S}$ in their corresponding roots (**root map**).

The map F is a bijection (onto its image) since it is a first return map of an invertible map. Since the flowers correspond to the first hyperbolic time for their respective roots, if we take any $x \in \gamma$, $\gamma \subset \mathcal{A}$, then we have $\|Df^n|E^{cs}(x)\| < \varsigma^n$, for all $n > 0$. This implies not only the fact that F restricted to any local stable leaf is a (uniform) contraction, but also that F has good hyperbolic properties (e.g., bounded distortion statements), as we will show in the next section.

Another important property of F is that we can decompose $\mathcal{A}$ into sets $\mathcal{A}_1, \ldots, \mathcal{A}_i, \ldots$, where each $\mathcal{A}_j$ is a union of local central manifolds, such that

$$F|_{\mathcal{A}_j} = f^j|_{\mathcal{A}_j}.$$

By our construction, each image $F(\mathcal{A}_j)$ crosses $\mathcal{A}$ in a Markovian way. This means that, if $F(\mathcal{A}_j)$ intersects the interior of a local unstable leaf, it contains the intersection of such a local leaf with $\mathcal{A}$.

Remark 2.20: Note that any $x \in \Lambda$ (remaining from the elimination algorithm) corresponds to one unique point $(x, -i), i > 0$ in the tower. So $f^i(x)$ belongs to $\mathcal{A}$. This implies that x has only negative Liapounov exponents in its central space. By [2, Th. A] (see also [10], [11]), f has some SRB measure μ_0. In the last section, we prove that it is Kolmogorov and the theorems we stated in the introduction of this paper.

3. An invariant measure for the root map

In this section we center our attention on systems as in the introduction admitting a Markov partition that extends to the ambient. This means that the Markov rectangles can be written as intersections of the attractor with open subsets of the ambient that also satisfy a Markov property. More precisely, we suppose that such open sets can be found admitting two transverse foliations: the center-stable foliation and an extension of the unstable foliation and these foliations map in a Markov fashion. In some cases, for instance, in systems derived from solenoid, we can use a not necessarily invariant foliation in place of an extension of the unstable foliation of the attractor, as long as such foliation satisfies three conditions that are stated in (p1), (p2) and (p3) below.

Under such hypotheses, we construct an F-invariant measure $\mu_{\mathcal{A}}$, which we will use, as well as some of its properties we prove here, to construct (in the last subsection of this section) the SRB-measure for the original diffeomorphism f. We use the methods of C. Liverani [8], in the dissipative version given in [13, section 4]. For the reader's convenience, we rewrite some of the propositions in these works, instead of just pointing out the necessary modifications.

Although we do not use it here, let us mention that $\mu_{\mathcal{A}}$ is, indeed, an SRB-measure for F. Moreover, $(F, \mu_{\mathcal{A}}, \mathcal{A})$ has exponential decay of correlations and satisfies the central limit theorem. Proofs of these facts can be found in [5].

For proving these results, there is no loss of generality in supposing that the extended Markov partition exhausts $\mathcal{M}$. We also remark that the tower construction that we did in the last section for Λ can be easily adapted for points of $\mathcal{M}$. We just have to see the stable manifolds as submanifolds of M (as in their classical meaning), not intersected with Λ.

3.1 Transfer operator. It is no restriction to suppose that the diameter of $\mathcal{M}$ is at most 1, and we do so in all that follows. Let m be the Lebesgue measure on $\mathcal{M}$. In this section and in the next, we will consider m normalized in such way that $m(\mathcal{A}) = 1$. Clearly, it is also no restriction to suppose that there is a unique good rectangle (just consider the other ones to be bad, replacing in the last section the value of the function κ on each of them by its greatest value).

We introduce linear operators

$$(U\varphi)(x) = \varphi(F(x))$$

and

$$(\mathbf{L}\,\varphi)(y) = \begin{cases} \varphi(F^{-1}(y))|\det DF(F^{-1}(y))|^{-1}, & \text{if } y \in F(\mathcal{A}), \\ 0, & \text{otherwise,} \end{cases}$$

where, if $F(x) = f^j(x)$, we have $DF(x) = D(f^j)(x)$. Observe that, changing variables $y = F(x)$,

$$\begin{aligned} \int_{\mathcal{A}} (\mathbf{L}\,\varphi)\psi\, dm &= \int_{F(\mathcal{A})} \frac{\varphi(F^{-1}(y))}{|\det DF(F^{-1}(y))|}\psi(y)dm(y) \\ &= \int_{\mathcal{A}} \varphi(x)\psi(F(x))dm(x) = \int_{\mathcal{A}} \varphi(U\psi)dm. \end{aligned}$$

We want to analyse the action of $\mathbf{L}$ on observable functions in terms of corresponding averages on local stable leaves. Of course, there is no canonical choice of probabilities supported on stable leaves. Instead, we average with respect to a whole class of measures, namely, the cone of Hölder continuous densities $\mathcal{D}(\gamma) = \mathcal{D}(a, \mu, \gamma)$, defined by

$$\mathcal{D}(\gamma) = \{\rho\colon \gamma \to \mathbb{R} \text{ such that } \rho(x) > 0 \forall x \in \gamma \text{ and } \log\rho \text{ is } (a,\mu)\text{-Hölder}\},$$

for each local stable leaf $\gamma \subset \mathcal{A}$. It is easy to check that $\mathcal{D}(\gamma)$ is a convex cone. Let $\theta = \theta_\gamma$ denote the corresponding projective metric. We also need a similar cone

$$\mathcal{D}_1(\gamma) = \mathcal{D}(a_1, \mu_1, \gamma),$$

corresponding to better Hölder constants $a_1 > 0$ and $0 < \mu_1 < 1$: we shall take $a \gg a_1 \gg 1$ and $0 < \mu < \mu_1 < 1$; cf. Lemma 3.2, (8), (13) below. Finally, we also consider the projective metric $\theta_+ = \theta_{+,\gamma}$ associated to the cone of positive densities

$$\mathcal{D}_+(\gamma) = \{\rho\colon \gamma \to \mathbb{R} \text{ such that } \rho > 0\}.$$

Given $\varphi\colon \mathcal{A} \to \mathbb{R}$ and $\rho \in \mathcal{D}(\gamma)$, we denote by $\int_\gamma \varphi\rho$ the integral of φ for the measure ρm_γ, where m_γ is the smooth measure induced on γ by the riemannian metric. We always suppose $\int_\gamma \rho = 1$, unless otherwise specified. Let $\varphi\colon \mathcal{A} \to \mathbb{R}$ and $\mathbf{L}\,\varphi$ be as defined above. Given any stable leaf γ, and $j \in \mathbb{N}$, let $\gamma_{j,k}$, $k = 1, \ldots, k_\gamma$ be the stable leaves such that $f^j(\gamma_{j,k}) = F(\gamma_{j,k}) \subset \gamma$. Then, for any $\rho \in \mathcal{D}(\gamma)$,

$$\begin{aligned} \int_\gamma (\mathbf{L}\,\varphi)\rho &= \sum_{j,k} \int_{F(\gamma_{j,k})} \frac{\varphi(F^{-1}(y))}{|\det DF(F^{-1}(y))|}\rho(y) \\ &= \sum_{j,k} \int_{\gamma_{j,k}} \varphi(x)\frac{\det(DF \mid \gamma_{j,k})(x)|}{|\det DF(x)|}\rho(F(x)) = \sum_{j,k} \int_{\gamma_{j,k}} \varphi\rho_{j,k}, \end{aligned}$$

where $F \mid \gamma_{j,k} = f^j \mid \gamma_{j,k}$, $(DF \mid \gamma_{j,k})$ denotes the restriction of DF to the tangent space of $\gamma_{j,k}$ and

$$\rho_{j,k} = \frac{|\det(DF \mid \gamma_{j,k})|}{|\det DF|}(\rho \circ F).$$

In the next lemmas we use the fact that local stable leaves form a continuous family of C^2 embedded submanifolds of $\mathcal{A}$, in particular, they have uniformly bounded curvature; see [13], Appendix A.

LEMMA 3.1: *There exists $K > 0$ which is a uniform Lipschitz constant for*

$$\log \frac{|\det(DF \mid \gamma_{j,k})|}{|\det DF|}$$

in $\gamma_{j,k}$, $\forall j, k$.

Proof: Let $\gamma^i_{j,k}$ be the leaf containing $f^i(\gamma_{j,k})$. Then, we will simply write Df^s_i instead of $Df \mid \gamma^i_{j,k}$, where f is our original diffeomorphism.

Given x, $y \in \gamma_{j,k}$, we have

$$\begin{aligned}
&\left| \log \frac{|\det(DF \mid \gamma_{j,k}(x)|}{|\det DF(x)|} - \log \frac{|\det(DF \mid \gamma_{j,k}(y)|}{|\det DF(y)|} \right| \\
&\leq \sum_{i=0}^{j-1} \left| \log \frac{|\det(Df^s_i)(f^i(x))|}{|\det Df(f^i(x))|} - \log \frac{|\det(Df^s_i(f^i(y))|}{|\det Df(f^i(y))|} \right|.
\end{aligned}$$

Since the curvature of the stable leaves is bounded, Df^s_i is globally Lipschitz and the expression above is less than or equal to

$$\sum_{i=0}^{j-1} K' d(f^i(x), f^i(y)) \leq K' \cdot \sum_{i=0}^{j-1} \varsigma^i \cdot d(x,y) \leq K \cdot d(x,y). \qquad \blacksquare$$

LEMMA 3.2: *There are $\lambda_1 < 1$ and $\tau_1 < 1$ such that, if a is large enough,*

(a) $\rho \in \mathcal{D}(\gamma) \Rightarrow \rho_{j,k} \in \mathcal{D}(\lambda_1 a, \mu, \gamma_{j,k})$ *for* $j > 0$,

(b) $\rho', \rho'' \in \mathcal{D}(\gamma) \Rightarrow \theta_{j,k}(\rho'_{j,k}, \rho''_{j,k}) \leq \tau_1 \theta(\rho', \rho'')$ *for* $j > 0$,

where θ and $\theta_{j,k}$ are the projective metrics associated to $\mathcal{D}(\gamma)$ and to $\mathcal{D}(\gamma_{j,k})$, respectively.

Proof: Clearly, $\rho > 0 \Rightarrow \rho_{j,k} > 0$.

Let $K > 0$ be the Lipschitz constant we found in the last lemma. We suppose

$$a > \frac{K}{1 - \varsigma^\mu},$$

where $\varsigma < 1$ is the same uniform bound for the contraction of F on stable leaves of the first section. Fix $\lambda_1 \in (\varsigma^\mu, 1)$ so that $a \geq K/(\lambda_1 - \varsigma^\mu)$. Then

$$\begin{aligned}
&|\log \rho_{j,k}(x) - \log \rho_{j,k}(y)| \leq |\log \rho(F(x)) - \log \rho(F(y))| \\
&+ \left|\log \frac{|\det(DF \mid \gamma_{j,k})(x)|}{|\det DF(x)|} - \log \frac{|\det(DF \mid \gamma_{j,k})(y)|}{|\det DF(y)|}\right. \\
&\leq a\, d(F(x), F(y))^\mu + K\, d(x,y) \leq (a\,(\varsigma^j)^\mu + K)\, d(x,y)^\mu \leq a\,\lambda_1\, d(x,y)^\mu.
\end{aligned}$$

This proves (a).

Now, in view of proposition 2.3 of [13], to prove (b) it suffices to show that $\mathcal{D}(\lambda_1 a, \mu, \gamma_{j,k})$ has finite diameter in $\mathcal{D}(\gamma_{j,k})$. Let $\theta_{+,j,k} = \theta_{+,\gamma_{j,k}}$ denote the projective metric associated to the cone $\mathcal{D}_+(\gamma_{j,k})$ of positive densities on $\gamma_{j,k}$. Observe that $\theta_{+,j,k}$ and $\theta_{j,k}$ are given by the expressions

$$\begin{aligned}
\theta_{+,j,k}(\varphi_1, \varphi_2) = \log \frac{\beta_{+,j,k}}{\alpha_{+,j,k}} &= \log \frac{\sup(\varphi_2/\varphi_1)}{\inf(\varphi_2/\varphi_1)} \\
&= \log \sup \left\{ \frac{\varphi_2(x)\varphi_1(y)}{\varphi_1(x)\varphi_2(y)} : x, y \in \gamma_{j,k} \right\}
\end{aligned}$$

and

$$\theta_{j,k}(\varphi_1, \varphi_2) = \log \frac{\beta_{j,k}(\varphi_1, \varphi_2)}{\alpha_{j,k}(\varphi_1, \varphi_2)},$$

where

$$\alpha_{j,k}(\varphi_1, \varphi_2) = \inf \left\{ \frac{\varphi_2(x)}{\varphi_1(x)}, \frac{\exp(ad(x,y)^\nu)\varphi_2(x) - \varphi_2(y)}{\exp(ad(x,y)^\nu)\varphi_1(x) - \varphi_1(y)} : x, y \in \gamma_{j,k}, x \neq y \right\},$$

and $\beta_{j,k}$ is given by a similar expression with sup instead of inf.

Given $\rho', \rho'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_{j,k})$ and $x, y \in \gamma_{j,k}$, we have

$$\frac{\exp(a\, d(x,y)^\mu) - \rho''(y)/\rho''(x)}{\exp(a\, d(x,y)^\mu) - \rho'(y)/\rho'(x)} \geq \frac{\exp(a\, d(x,y)^\mu) - \exp(\lambda_1 a\, d(x,y)^\mu)}{\exp(a\, d(x,y)^\mu) - \exp(-\lambda_1 a\, d(x,y)^\mu)} \geq \tau_1'$$

where $\tau_1' = \inf\{(z - z^{\lambda_1})/(z - z^{-\lambda_1}) : z > 1\} \in (0,1)$. Therefore,

$$\alpha_{j,k}(\rho', \rho'') \geq \tau_1'\, \alpha_{+,j,k}(\rho', \rho'').$$

In just the same way, one finds $\tau_2' > 1$ such that $\beta_{j,k}(\rho', \rho'') \leq \tau_2' \beta_{+,j,k}(\rho', \rho'')$. Thus,

$$\theta_{j,k}(\rho', \rho'') \leq \theta_{+,j,k}(\rho', \rho'') + \log(\tau_2'/\tau_1')$$

for every $\rho', \rho'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_{j,k})$.

Next we find a uniform upper bound for $\theta_{+,j}(\rho',\rho'')$ with $\rho',\rho''\in\mathcal{D}(\gamma_{j,k})$. We normalize $\int_{\gamma_{j,k}}\rho' = 1 = \int_{\gamma_{j,k}}\rho''$ and then the mean value theorem gives

$$\frac{\rho''}{\rho'}(x) \geq \frac{\exp(-a\,(\operatorname{diam}\gamma_{j,k})^{\mu})}{\exp(a\,(diam\gamma_{j,k})^{\mu})} \geq \exp(-2a).$$

Recall that we suppose $\operatorname{diam}(\mathcal{M}) \leq 1$. It follows that

$$\alpha_{+,j,k}(\rho',\rho'') \geq \exp(-2a) \quad \text{and} \quad \beta_{+,j,k}(\rho',\rho'') \leq \exp(2a)$$

and so $\theta_{+,j,k}(\rho',\rho'') \leq 4a$, for all $\rho',\rho''\in\mathcal{D}(\gamma_{j,k})$. Altogether, we have shown that the $\theta_{j,k}$-diameter of $\mathcal{D}(\lambda_1 a,\mu,\gamma_{j,k})$ is bounded by $4a+\log(\tau_2'/\tau_1')$. ∎

Now let us consider a family of C^2 diffeomorphisms

$$\pi = \pi(\tilde{\gamma},\gamma)\colon \tilde{\gamma} \to \gamma,$$

C^2-close to the inclusion map of $\tilde{\gamma}$ in M, chosen satisfying (p1), (p2) and (p3) below, which is always possible (see [13, section 3], and references therein), if we take it near to the (f-invariant) projection along unstable leaves. We observe that such C^2 projections are not necessarily f- or F-invariant.

As $\gamma_{j,k}$'s are the connected components of $F^{-1}(\gamma)$ (such that $F|_{\gamma_{j,k}} = f^j|_{\gamma_{j,k}}$) we put $\tilde{\gamma}_{j,k}$'s to be the connected components of $F^{-1}(\tilde{\gamma})$ (numbered in such a way that $\tilde{\gamma}_{j,k}$ is close to $\gamma_{j,k}$ and both components are in the same rectangle). We still write $\pi_{j,k} = \pi(\tilde{\gamma}_{j,k},\gamma_{j,k})$. Let $d(y',y'')$ denote the distance between two points $y',y''\in\mathcal{A}$ belonging to the same unstable leaf Γ, measured along Γ. Then there are constants $a_0>0$, $\nu_0>0$, $\lambda_u<1$, depending only on f, such that
(p1) π and $\log|\det D\pi|$ are a_0-Lipschitz maps;
(p2) $\log|\det D\pi(y)| \leq a_0\, d(y,\pi(y))^{\nu_0}$ for every $y\in\tilde{\gamma}$;
(p3) $d(x,\pi_{j,k}(x)) \leq \lambda_u\, d(F(x),\pi F(x))$ for every $x\in\tilde{\gamma}_{j,k}$.

For any γ, $\tilde{\gamma}$ as before, we define the **distance** between γ and $\tilde{\gamma}$ by

$$d(\gamma,\tilde{\gamma}) = \sup\{d(y,\pi(y)) : y\in\tilde{\gamma}\}.$$

As a direct consequence of (p3), the map induced by F in the space of local stable leaves is expanding for this distance:

$$d(\gamma,\tilde{\gamma}) \geq \lambda_u^{-1}\, d(\gamma_{j,k},\tilde{\gamma}_{j,k}), \quad \text{for every } \gamma,\tilde{\gamma}. \tag{5}$$

Next, to every $\rho\in\mathcal{D}_1(\gamma)$ we associate the density $\tilde{\rho}\colon\tilde{\gamma}\to\mathbb{R}$ defined by

$$\tilde{\rho}(y) = \rho(\pi(y))\cdot|\det D\pi(y)|. \tag{6}$$

Clearly, $\tilde\rho > 0$. Moreover, as a consequence of (p1), $\log\tilde\rho = \log\rho\circ\pi + \log|\det D\pi|$ is $(\tilde a_1, \mu_1)$-Hölder continuous with $\tilde a_1 = a_1 a_0^{\mu_1} + a_0$. Recall that $\operatorname{diam}(\mathcal{M}) \le 1$. This also implies that $\log\tilde\rho$ is $(\tilde a, \mu)$-Hölder continuous, for $\tilde a = a_1 a_0^{\mu_1} + a_0$. We suppose

$$a/2 \ge \tilde a = a_1 a_0^{\mu_1} + a_0, \tag{7}$$

so that, in particular, $\rho \in \mathcal{D}_1(\gamma) \Rightarrow \tilde\rho \in \mathcal{D}(\tilde\gamma)$. (This step is one of the reasons why we need the auxiliary cone $\mathcal{D}_1(\gamma)$, the other one is in the proof of Lemma 3.5.) Note also that $\int_\gamma \rho = \int_{\tilde\gamma} \tilde\rho$, by change of variables.

3.2 An invariant cone for the transfer operator. Now we introduce a cone of functions that, as we shall see, is strictly invariant under the transfer operator $\mathbf{L}$. This fact is at the basis of our construction of $\mu_{\mathcal{A}}$.

Given $b > 0$, $c > 0$, and $\nu \in (0,1]$, we let $\mathcal{C}(b,c,\nu)$ be the cone of bounded functions $\varphi\colon \mathcal{A} \to \mathbb{R}$ satisfying conditions (A), (B) and (C) below:

(A) $\int_\gamma \varphi\rho > 0$ for every $\gamma \in \mathcal{F}^s_{loc}(\mathcal{A})$ and every $\rho \in \mathcal{D}(\gamma)$;

(B) the map $\mathcal{D}(\gamma) \ni \rho \mapsto \log\int_\gamma \varphi\rho$ is b-Lipschitz, that is,

$$\left|\log\int_\gamma \varphi\rho' - \log\int_\gamma \varphi\rho''\right| \le b\,\theta(\rho',\rho'')$$

for every $\rho', \rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$, and every $\gamma \in \mathcal{F}^s_{loc}$;

(C) the map $\mathcal{F}^s_{loc} \ni \gamma \mapsto \int_\gamma \varphi\rho$ is (c,ν)-Hölder, more precisely,

$$\left|\log\int_\gamma \varphi\rho - \log\int_{\tilde\gamma} \varphi\tilde\rho\right| \le c\, d(\gamma,\tilde\gamma)^\nu$$

for every $\rho \in \mathcal{D}_1(\gamma)$ and every pair $\gamma, \tilde\gamma \in \mathcal{F}^s_{loc}(\mathcal{A})$.

As a matter of fact, we want to think of the elements of $\mathcal{C}(b,c,\nu)$ as equivalence classes of bounded functions for the equivalence relation

$$\varphi_1 \sim \varphi_2 \quad\Leftrightarrow\quad \varphi_1|\gamma = \varphi_2|\gamma \quad m_\gamma\text{-almost everywhere, for every } \gamma \in \mathcal{F}^s_{loc}.$$

However, replacing an equivalence class by any of its members never results in ambiguity, and so we ignore this formal distinction, in order not to overload the notations.

Let us also note that (B) is automatically satisfied (with $b = 1$) in the particular case when φ is nonnegative:

$$\frac{\int_\gamma \varphi\rho'}{\int_\gamma \varphi\rho''} \le \frac{\sup\rho'}{\inf\rho''} \le \frac{\sup\rho'/\inf\rho'}{\inf\rho''/\sup\rho''} = \exp(\theta_+(\rho',\rho'')) \le \exp(\theta(\rho',\rho'')). \tag{8}$$

In the sequel we take b and c large, and ν close to zero; cf. Proposition 3.4 and (12).

LEMMA 3.3: *$\mathcal{C}(b,c,\nu)$ is a convex cone with $-\overline{\mathcal{C}(b,c,\nu)} \cap \overline{\mathcal{C}(b,c,\nu)} = \{0\}$.*

Proof: See [13, section 4] for the proof, which follows through standard arguments. ∎

Calculating the projective metric $\Theta = \Theta_{b,c,\nu} = \log(\beta/\alpha)$ associated to $\mathcal{C}(b,c,\nu)$, we obtain:

$$\alpha(\varphi_1,\varphi_2) = \inf\Big\{\frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'}, \frac{\int_\gamma \varphi_2\rho'}{\int_\gamma \varphi_1\rho'}\xi(\rho',\rho'',\varphi_1,\varphi_2),$$
$$\frac{\int_\gamma \varphi_2\rho}{\int_\gamma \varphi_1\rho}\eta(\rho,\tilde\rho,\varphi_1,\varphi_2), \frac{\int_{\tilde\gamma} \varphi_2\tilde\rho}{\int_{\tilde\gamma} \varphi_1\tilde\rho}\eta(\tilde\rho,\rho,\varphi_1,\varphi_2)\Big\}$$

where the infimum runs over all $\rho' \in \mathcal{D}(\gamma)$, $\rho'' \in \mathcal{D}(\gamma)$, $\rho \in \mathcal{D}_1(\gamma)$, and every pair of local stable leaves γ and $\tilde\gamma$; $\beta(\varphi_1,\varphi_2)$ is given by a similar expression, with inf replaced by sup. The explicit expressions of ξ and η above are just (respectively)

$$\frac{\exp(b\theta(\rho',\rho'')) - (\int_\gamma \varphi_2\rho''/\int_\gamma \varphi_2\rho')}{\exp(b\theta(\rho',\rho'')) - (\int_\gamma \varphi_1\rho''/\int_\gamma \varphi_1\rho')} \quad\text{and}\quad \frac{\exp(c\,d(\gamma,\tilde\gamma)^\nu) - (\int_{\tilde\gamma} \varphi_2\tilde\rho/\int_\gamma \varphi_2\rho)}{\exp(c\,d(\gamma,\tilde\gamma)^\nu) - (\int_{\tilde\gamma} \varphi_1\tilde\rho/\int_\gamma \varphi_1\rho)}.$$

PROPOSITION 3.4 (Cone invariance): *There exists $\lambda_2 < 1$ so that $\mathbf{L}(\mathcal{C}(b,c,\nu)) \subset \mathcal{C}(\lambda_2 b, \lambda_2 c, \nu)$ for every large enough b and c.*

Proof: Let $\rho \in \mathcal{D}(\gamma)$ and $\rho_{j,k}$ be as defined above. Lemma 3.2(a) ensures that $\rho_{j,k} \in \mathcal{D}(\gamma_{j,k})$ and so $\int_{\gamma_{j,k}} \varphi\rho_{j,k} > 0$ for each pair j,k. As a consequence, $\int_\gamma (\mathbf{L}\,\varphi)\rho = \sum_{j,k}\int \varphi\rho_{j,k} > 0$, which proves the invariance of (A).

To prove the invariance of condition (B), let $\rho',\rho'' \in \mathcal{D}(\gamma)$ with $\int_\gamma \rho' = 1 = \int_\gamma \rho''$. Denote

$$\rho'_{j,k} = \frac{|\det(DF \mid \gamma_{j,k})|}{|\det DF|}(\rho' \circ F) \quad\text{and}\quad \rho''_{j,k} = \frac{|\det(DF \mid \gamma_{j,k})|}{|\det DF|}(\rho'' \circ F).$$

Moreover, let $\rho^-_{j,k} = \rho'_{j,k}/\int_{\gamma_{j,k}} \rho'_{j,k}$ and $\rho^=_{j,k} = \rho''_{j,k}/\int_{\gamma_{j,k}} \rho''_{j,k}$. Then, using condition (B) for φ, followed by Lemma 3.2(b),

$$\int_\gamma (\mathbf{L}\,\varphi)\rho'' = \sum_{j,k}\int_{\gamma_{j,k}} \varphi\rho''_{j,k} = \sum_{j,k}\int_{\gamma_{j,k}} \rho''_{j,k}\int_{\gamma_{j,k}} \varphi\rho^=_{j,k}$$
$$\le \sum_{j,k}\int_{\gamma_{j,k}} \rho''_{j,k}\cdot\exp(b\theta(\rho^-_{j,k},\rho^=_{j,k}))\int_{\gamma_{j,k}} \varphi\rho^-_{j,k}$$

$$\leq \sum_{j,k} \exp(b\theta(\rho'_{j,k}, \rho''_{j,k})) \frac{\int_{\gamma_{j,k}} \rho''_{j,k}}{\int_{\gamma_{j,k}} \rho'_{j,k}} \int_{\gamma_{j,k}} \varphi\rho'_{j,k}$$

$$\leq \exp(b\tau_1\theta(\rho', \rho'')) \sum_{j,k} \frac{\int_{\gamma_{j,k}} \rho''_{j,k}}{\int_{\gamma_{j,k}} \rho'_{j,k}} \int_{\gamma_{j,k}} \varphi\rho'_{j,k}.$$

By the same arguments as in Lemma 3.2,

$$\frac{\rho''_{j,k}}{\rho'_{j,k}}(x) = \frac{\rho''(F(x))}{\rho'(F(x))} \leq \exp(\theta(\rho', \rho'')) \leq \exp(\theta_+(\rho', \rho'')).$$

Thus, replacing above we conclude that

$$\int_\gamma (\mathbf{L}\,\varphi)\rho'' \leq \exp(b\tau_1\theta(\rho', \rho'') + \theta(\rho', \rho'')) \sum_{j,k} \int_{\gamma_{j,k}} \varphi\rho'_{j,k}$$

$$\leq \exp(b\lambda_2\theta(\rho', \rho'')) \int_\gamma (\mathbf{L}\,\varphi)\rho',$$

as long as we fix $\lambda_2 \in (\tau_1, 1)$ and suppose $b \geq 1/(\lambda_2 - \tau_1)$.

Now we prove invariance of condition (C).

Given two stable leaves γ and $\tilde{\gamma}$, let $\rho \in \mathcal{D}_1(\gamma) \subset \mathcal{D}(\gamma)$. As we have seen before,

$$\int_\gamma (\mathbf{L}\,\varphi)\rho = \sum_{j,k} \int_{\gamma_{j,k}} \varphi\rho_{j,k} \quad \text{and} \quad \int_{\tilde{\gamma}} (\mathbf{L}\,\varphi)\tilde{\rho} = \sum_{j,k} \int_{\tilde{\gamma}_{j,k}} \varphi(\tilde{\rho})_{j,k},$$

where $(\tilde{\rho})_{j,k}\colon \tilde{\gamma}_{j,k} \to \mathbb{R}$ is defined by

$$(\tilde{\rho})_{j,k}(x) = \rho(\pi F(x))|\det D\pi(F(x))| \frac{|\det(DF \mid \tilde{\gamma}_{j,k})(x)|}{|\det DF(x)|}.$$

Since $\rho_{j,k} \in \mathcal{D}_1(\gamma_{j,k})$, we may invoke property (C) for φ together with (6), to conclude that

$$\left| \log \int_{\gamma_{j,k}} \varphi\rho_{j,k} - \log \int_{\tilde{\gamma}_{j,k}} \varphi\widetilde{\rho_{j,k}} \right| \leq c\, d(\gamma_{j,k}, \tilde{\gamma}_{j,k})^\nu \leq c\, \lambda_u^\nu\, d(\gamma, \tilde{\gamma})^\nu,$$

where

$$\widetilde{\rho_{j,k}}(x) = \rho(F\pi_{j,k}(x)) \frac{|\det(DF \mid \gamma_{j,k})(\pi_{j,k}(x))|}{|\det DF(\pi_{j,k}(x))|} |\det D\pi_{j,k}(x)|$$

with $\pi_{j,k} = \pi(\tilde{\gamma}_{j,k}, \gamma_{j,k})$. Next, we use the following estimate, which is given by the auxiliary lemma 3.5 below:

$$\left| \log \int_{\tilde{\gamma}_{j,k}} \varphi\widetilde{\rho_{j,k}} - \log \int_{\tilde{\gamma}_{j,k}} \varphi(\tilde{\rho})_{j,k} \right| \leq K_0\, d(\gamma, \tilde{\gamma})^\nu,$$

for some constant $K_0 > 0$ that does not depend on c. Altogether,

$$\left|\log \int_{\gamma_{j,k}} \varphi\rho_{j,k} - \log \int_{\tilde{\gamma}_{j,k}} \varphi(\tilde{\rho})_{j,k}\right| \leq (c\,\lambda_u^\nu + K_0)\, d(\gamma,\tilde{\gamma})^\nu,$$

and so

$$\left|\log \int_{\gamma} (\mathbf{L}\,\varphi)\rho - \log \int_{\tilde{\gamma}} (\mathbf{L}\,\varphi)\tilde{\rho}\right| \leq (c\,\lambda_u^\nu + K_0)\, d(\gamma,\tilde{\gamma})^\nu \leq \lambda_2 c\, d(\gamma,\tilde{\gamma})^\nu,$$

as long as we take $\lambda_2 \in (\lambda_u^\nu, 1)$ and suppose $c \geq K_0/(\lambda_2 - \lambda_u^\nu)$. ∎

In what follows, we fix j,k so we drop k in our notations (e.g., we write γ_j instead of $\gamma_{j,k}$), for the sake of simplicity.

We have reduced the proof of the last proposition to proving the following auxiliary statement. We note that it is not necessary in the case we have an f-invariant ambient unstable foliation (for instance, in the case derived from Anosov systems).

LEMMA 3.5: *There is $K_0 > 0$ depending only on f, a, a_1, b, such that*

$$\left|\log \int_{\tilde{\gamma}_j} \varphi\widetilde{\rho}_j - \log \int_{\tilde{\gamma}_j} \varphi(\tilde{\rho})_j\right| \leq K_0\, d(\gamma,\tilde{\gamma})^\nu$$

for every $\varphi \in \mathcal{C}(b,c,\nu)$ and every $\gamma,\tilde{\gamma} \in \mathcal{F}^s_{loc}$, $\rho \in \mathcal{D}_1(\gamma)$, and j.

Proof: We use $K_1,\ldots,K_6$ to denote sufficiently large constants, depending only on f, a, a_1, b. The previous arguments prove that

(i)
$$\begin{aligned}\rho \in \mathcal{D}_1(\gamma) &\Rightarrow \tilde{\rho} \in \mathcal{D}(\tilde{a}_1,\mu_1,\tilde{\gamma})\\ &\Rightarrow \rho' = (\tilde{\rho})_j \in \mathcal{D}(\lambda_1\tilde{a}_1,\mu_1,\tilde{\gamma}_j) \subset \mathcal{D}(\tilde{a}_1,\mu_1,\tilde{\gamma}_j),\end{aligned}$$

(ii)
$$\begin{aligned}\rho \in \mathcal{D}_1(\gamma) &\Rightarrow \rho_j \in \mathcal{D}(\lambda_1 a_1,\mu_1,\gamma_j) \subset \mathcal{D}(a_1,\mu_1,\gamma_j)\\ &\Rightarrow \rho'' = \widetilde{\rho}_j \in \mathcal{D}(\tilde{a}_1,\mu_1,\tilde{\gamma}_j),\end{aligned}$$

where $\tilde{a}_1 = a_1 a_0^{\mu_1} + a_0$. In (i), we suppose that a_1 is large enough so that Lemma 3.2(a) holds with the cone $\mathcal{D}(\tilde{a}_1,\mu_1,\gamma)$ in the place of $\mathcal{D}(\gamma) = \mathcal{D}(a,\mu,\gamma)$.

It follows from (i) and (ii) that both ρ' and ρ'' belong to $\mathcal{D}(\tilde{a}_1,\mu,\tilde{\gamma}_j)$ which, by (7), is contained in $\mathcal{D}(a/2,\mu,\tilde{\gamma}_j) \subset \mathcal{D}(\tilde{\gamma}_j)$. Hence, using condition (B) for the normalized densities $\rho'/\int_{\tilde{\gamma}_j}\rho'$ and $\rho''/\int_{\tilde{\gamma}_j}\rho''$,

$$\left|\log \int_{\tilde{\gamma}_j} \varphi\rho' - \log \int_{\tilde{\gamma}_j} \varphi\rho''\right| \leq b\theta_j(\rho',\rho'') + \left|\log \int_{\tilde{\gamma}_j} \rho' - \log \int_{\tilde{\gamma}_j} \rho''\right|. \tag{9}$$

In order to bound the two terms on the right hand side, let us take a look at the relation

(10)
$$\frac{\rho'(x)}{\rho''(x)} = \frac{\rho(\pi f^j(x))}{\rho(f^j\pi_j(x))}\frac{|\det D\pi(f^j(x))|}{|\det D\pi_j(x)|}\frac{|\det(Df^j \mid \tilde{\gamma}_j)(x)|}{|\det(Df^j \mid \gamma_j)(\pi_j(x))|}\frac{|\det Df^j(\pi_j(x))|}{|\det Df^j(x)|}.$$

First, we need to examine $d(\pi f^j(x), f^j(\pi_j(x)))$.

Let Γ_0 be a leaf of the foliation we used to define π_j, passing by x, and let $\xi_0 \subset \Gamma_0$ be a curve joining x to $\pi_j(x)$, with length$(\xi_0) = d(x, \pi_j(x))$. The key remark is that the angle of each iterate $f^i(\Gamma_0)$, $i \geq 0$ with the direction of the foliation we just mentioned is bounded, in each point, by some constant $H > 0$, which depends only on f. Then,

$$d(\pi(f^j(x)), f^j(\pi_j(x))) \leq H \cdot d(\gamma, \tilde{\gamma})$$

and we have

$$d(f^j(x), f^j(\pi_j(x))) = \text{length}(f^j(\xi_0)) \leq (1+H)\cdot d(\gamma, \tilde{\gamma}).$$

More generally,

$$\text{(11)} \qquad \underset{1<i\leq j}{\text{length}}(f^i(\xi_0)) \leq (1+H)\lambda_u^{j-i} d(\gamma, \tilde{\gamma}),$$

a relation that we will use to obtain bounded distortion for $\|\det Df^j\|$; that is, due to the equation above, we have

$$\|\log|\det Df^j(x)| - \log|\det Df^j(\pi_j(x))|\| \leq K_2 d(\gamma, \tilde{\gamma}).$$

This is because

$$\|\log|\det Df^j(x)| - \log|\det Df^j(\pi_j(x))|\| \leq \sum_{i=0}^{j-1} K_1 \cdot \text{length}(f^i(\xi_0))$$
$$\leq K_1\cdot(1+H)\cdot\sum_{i=0}^{j-1}\lambda_u^{j-i}\cdot d(\gamma,\tilde{\gamma}) \leq K_1\cdot(1+H)\cdot\sum_{i=0}^{\infty}\lambda_u^{i}\cdot d(\gamma,\tilde{\gamma}) \leq K_2\cdot d(\gamma,\tilde{\gamma})$$

where K_1 is a bound for the Lipschitz constant of $\log\|\det Df\|$. Combining with the assumption $\rho \in \mathcal{D}_1(\gamma)$, we find

$$\left|\log\rho(\pi f^j(x)) - \log\rho(f^j\pi_j(x))\right| \leq a_1 H^{\mu_1}\cdot d(\gamma,\tilde{\gamma})^{\mu_1}.$$

On the other hand, by (p2),

$$\left|\log|\det D\pi(f^j(x))| - \log|\det D\pi_j(x)|\right|$$
$$\leq a_0\, d(f^j(x), \pi f^j(x))^{\nu_0} + a_0\, d(x, \pi_j(x))^{\nu_0} \leq 2a_0 d(\gamma,\tilde{\gamma})^{\nu_0}.$$

Next, using also the Hölder property of the tangent bundle to $\mathcal{F}^s_{loc}$, in the same manner as above for $\log|\det(Df^j)|$, we have

$$\big|\log|\det(Df^j \mid \tilde{\gamma}_j)(x)| - \log|\det(Df^j \mid \gamma_j)(\pi_j(x))|\big| \leq K_3\, d(\gamma,\tilde{\gamma})^{\nu_0}.$$

At this point we assume that

$$0 < \mu < \mu + \nu \leq \mu_1 \leq \nu_0. \tag{12}$$

Then, replacing the previous bounds in (10),

$$\big|\log\rho'(x) - \log\rho''(x)\big| \leq K_4 d(\gamma,\tilde{\gamma})^{\mu_1} \tag{13}$$

for some sufficiently large $K_4 > 0$, and every $x \in \tilde{\gamma}_j$ and j. In particular,

$$\left|\log\int_{\tilde{\gamma}_j}\rho' - \log\int_{\tilde{\gamma}_j}\rho''\right| \leq K_4 d(\gamma,\tilde{\gamma})^{\mu_1} \tag{14}$$

and

$$\theta_{+,j}(\rho',\rho'') = \log\left(\frac{\sup_{\tilde{\gamma}_j}(\rho''/\rho')}{\inf_{\tilde{\gamma}_j}(\rho''/\rho')}\right) \leq 2K_4\, d(\gamma,\tilde{\gamma})^{\mu_1}. \tag{15}$$

Inequality (14) provides the kind of bound we want for the last term in (9).

To bound the term $b\theta_j(\rho',\rho'')$, we combine (15) with the relation (recall, e.g., the proof of Lemma 3.2)

$$\theta_j(\rho',\rho'') \leq \theta_{+,j}(\rho',\rho'') + \log(\hat{\tau}_2'/\hat{\tau}_1'), \tag{16}$$

where

$$\hat{\tau}_1' = \inf\left\{\frac{\exp(a\, d(x,y)^\mu) - \rho''(y)/\rho''(x)}{\exp(a\, d(x,y)^\mu) - \rho'(y)/\rho'(x)} : x, y \in \tilde{\gamma}_j \text{ with } x \neq y\right\}$$

and $\hat{\tau}_2'$ is given by a similar expression, with inf replaced by sup. All that is left to do is to bound $|\log\hat{\tau}_1'|$ and $|\log\hat{\tau}_2'|$. Let us denote

$$B' = (\rho'(y)/\rho'(x))\exp(-a\, d(x,y)^\mu) \quad\text{and}\quad B'' = (\rho''(y)/\rho''(x))\exp(-a\, d(x,y)^\mu).$$

Clearly, $\rho' \in \mathcal{D}(a/2,\mu,\tilde{\gamma}_j)$ implies $\log B' \leq -(a/2)d(x,y)^\mu < 0$, and analogously for ρ'' and B''. In particular,

$$|B' - B''| \leq |\log B' - \log B''| = |\log\rho'(y) - \log\rho'(x) - \log\rho''(y) + \log\rho''(x)|.$$

On the one hand, (13) implies

$$
\text{(17)} \qquad \begin{aligned} |B' - B''| &\le |\log \rho'(y) - \log \rho''(y)| + |\log \rho'(x) - \log \rho''(x)| \\ &\le 2K_4 d(\gamma, \tilde{\gamma})^{\mu}. \end{aligned}
$$

On the other hand,

$$
\text{(18)} \qquad \begin{aligned} |B' - B''| &\le |\log \rho'(y) - \log \rho'(x)| + |\log \rho''(y) - \log \rho''(x)| \\ &\le 2\tilde{a}_1\, d(x,y)^{\mu_1}, \end{aligned}
$$

because $\rho', \rho'' \in \mathcal{D}(\tilde{a}_1, \mu_1, \tilde{\gamma}_j)$. Since we are taking $\mu_1 \ge \mu + \nu$, it follows that

$$
|B' - B''| \le K_5\, d(x,y)^{\mu}\, d(\gamma, \tilde{\gamma})^{\nu},
$$

as long as $K_5 \ge \max\{2K_4, 2\tilde{a}_1\}$. Indeed, this last inequality is a direct consequence of (17) if $d(x,y) \ge d(\gamma, \tilde{\gamma})$, and of (18) in the case when $d(x,y) \le d(\gamma,\tilde{\gamma})$. Then,

$$
\log \frac{1 - B''}{1 - B'} \le \frac{|B' - B''|}{1 - \max\{B', B''\}} \le \frac{K_5\, d(x,y)^{\mu}\, d(\gamma,\tilde{\gamma})^{\nu}}{1 - \exp(-(a/2)\, d(x,y)^{\mu})} \le K_6\, d(\gamma, \tilde{\gamma})^{\nu}.
$$

Replacing in $\hat{\tau}_1'$ and $\hat{\tau}_2'$, we find

$$
\log \hat{\tau}_1' \ge -K_6\, d(\gamma, \tilde{\gamma})^{\nu} \quad \text{and} \quad \log \hat{\tau}_2' \le K_6\, d(\gamma, \tilde{\gamma})^{\nu},
$$

and so, in view of (15) and (16),

$$
\theta_j(\rho', \rho'') \le 2K_4\, d(\gamma, \tilde{\gamma})^{\mu_1} + 2K_6\, d(\gamma, \tilde{\gamma})^{\nu}.
$$

Finally, by (9) and (14),

$$
\left| \log \int_{\tilde{\gamma}_j} \varphi \rho' - \log \int_{\tilde{\gamma}_j} \varphi \rho'' \right| \le (2b+1)\, K_4\, d(\gamma, \tilde{\gamma})^{\mu_1} + 2bK_6\, d(\gamma, \tilde{\gamma})^{\nu} \le K_0\, d(\gamma, \tilde{\gamma})^{\nu}
$$

for some $K_0 > 0$, which concludes the proof of the lemma. ∎

3.3 Contraction of the projective metric. In the proof of the next result we use the projective metric Θ_+ associated to the cone of bounded functions satisfying $\int_\gamma \varphi\rho > 0$ for every γ and every $\rho \in \mathcal{D}(\gamma)$. In the same way as we calculated Θ, one checks that $\Theta_+(\varphi_1, \varphi_2) = \log(\beta_+(\varphi_1,\varphi_2)/\alpha_+(\varphi_1,\varphi_2))$, with

$$
\alpha_+(\varphi_1, \varphi_2) = \inf_{\rho,\gamma} \left\{ \frac{\int_\gamma \varphi_2 \rho}{\int_\gamma \varphi_1 \rho} \right\} \quad \text{and} \quad \beta_+(\varphi_1, \varphi_2) = \sup_{\rho,\gamma} \left\{ \frac{\int_\gamma \varphi_2 \rho}{\int_\gamma \varphi_1 \rho} \right\}
$$

(taken over every $\rho \in \mathcal{D}(\gamma)$ and every stable leaf γ).

PROPOSITION 3.6 (finite diameter): *For $b > 0$, $c > 0$, $\nu \in (0,1]$, the Θ-diameter $D_2 = \sup\{\Theta(\mathbf{L}\,\varphi_1, \mathbf{L}\,\varphi_2) : \varphi_1, \varphi_2 \in \mathcal{C}(b,c,\nu)\}$ of $\mathbf{L}(\mathcal{C}(b,c,\nu))$ is finite.*

Proof: As in the proof of Lemma 3.2, the argument has two parts. In the first step, we bound D_2 in terms of the Θ_+-diameter of $\mathbf{L}(\mathcal{C}(b,c,\nu))$. Let $\varphi_1, \varphi_2 \in \mathcal{C}(\lambda_2 b, \lambda_2 c, \nu)$.

Given $\rho', \rho'' \in \mathcal{D}(\gamma)$ and $\rho \in \mathcal{D}_1(\gamma)$, we have

$$\xi(\rho', \rho'', \varphi_1, \varphi_2) \geq \frac{\exp(b\theta(\rho', \rho'')) - \exp(b\lambda_2\,\theta(\rho', \rho''))}{\exp(b\theta(\rho', \rho'')) - \exp(-b\lambda_2\,\theta(\rho', \rho''))} \geq \tau_1'$$

where $\tau_1' = \inf\{(z - z^{\lambda_2})/(z - z^{-\lambda_2}) : z > 1\} \in (0,1)$. In just the same way,

$$\xi(\rho', \rho'', \varphi_1, \varphi_2) \leq \tau_2', \quad \eta(\rho', \rho'', \varphi_1, \varphi_2) \in [\tau_1', \tau_2'], \quad \eta(\tilde{\rho}, \rho, \varphi_1, \varphi_2) \in [\tau_1', \tau_2'],$$

where $\tau_2' = \sup\{(z - z^{-\lambda_2})/(z - z^{-\lambda_2}) : z > 1\} \in (1, +\infty)$. As a direct consequence, $\alpha(\varphi_1, \varphi_2) \geq \tau_1'\,\alpha_+(\varphi_1, \varphi_2)$ and $\beta(\varphi_1, \varphi_2) \leq \tau_2'\,\beta_+(\varphi_1, \varphi_2)$, and so

$$\Theta(\varphi_1, \varphi_2) \leq \Theta_+(\varphi_1, \varphi_2) + \log(\tau_2'/\tau_1')$$

for all $\varphi_1, \varphi_2 \in \mathcal{C}(\lambda_2 b, \lambda_2 c, \nu) \supset \mathbf{L}(\mathcal{C}(b,c,\nu))$.

Now we present the second and last step, where we show that the Θ_+-diameter of $\mathbf{L}(\mathcal{C}(b,c,\nu))$ is finite, that is, there is a uniform upper bound for

$$\frac{\int_{\gamma''}(\mathbf{L}\,\varphi_2)\rho''}{\int_{\gamma''}(\mathbf{L}\,\varphi_1)\rho''} \Big/ \frac{\int_{\gamma'}(\mathbf{L}\,\varphi_2)\rho'}{\int_{\gamma'}(\mathbf{L}\,\varphi_1)\rho'}, \quad \varphi_1, \varphi_2 \in \mathcal{C}(b,c,\nu),\ \rho' \in \mathcal{D}(\gamma'),\ \rho'' \in \mathcal{D}(\gamma'').$$

(We continue writing j instead of j,k and l instead of l,m so as not to overload the notations)

In fact, we have

$$\frac{\int_{\gamma''}(\mathbf{L}\,\varphi_2)\rho''}{\int_{\gamma''}(\mathbf{L}\,\varphi_1)\rho''} \cdot \frac{\int_{\gamma'}(\mathbf{L}\,\varphi_1)\rho'}{\int_{\gamma'}(\mathbf{L}\,\varphi_2)\rho'} = \frac{\sum_{j,l}\left(\int_{\gamma_j''} \varphi_2 \rho_j'' \cdot \int_{\gamma_l'} \varphi_1 \rho_l'\right)}{\sum_{j,l}\left(\int_{\gamma_j''} \varphi_1 \rho_j'' \cdot \int_{\gamma_l'} \varphi_2 \rho_l'\right)}.$$

So all we have to do is to bound uniformly each positive part

$$\frac{\left(\int_{\gamma_j''} \varphi_2 \rho_j'' \cdot \int_{\gamma_l'} \varphi_1 \rho_l'\right)}{\left(\int_{\gamma_j''} \varphi_1 \rho_j'' \cdot \int_{\gamma_l'} \varphi_2 \rho_l'\right)} = \frac{\left(\int_{\gamma_j} \varphi_2 \overline{\overline{\rho}}_j \cdot \int_{\gamma_l'} \varphi_1 \overline{\rho}_l\right)}{\left(\int_{\gamma_j''} \varphi_1 \overline{\overline{\rho}}_j \cdot \int_{\gamma_l'} \varphi_2 \overline{\rho}_l\right)},$$

where the bars mean normalization of the densities. Recall also that $\rho_j' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j')$ and $\rho_j'' \in \mathcal{D}(\lambda_1 a, \mu, \gamma_j'')$, by Lemma 3.2. Hence our claim that the quotient in the equation above is uniformly bounded will follow if we show that

$$\sup \frac{\int_{\gamma_2} \varphi\,\rho_2}{\int_{\gamma_1} \varphi\,\rho_1} < +\infty \tag{19}$$

where the supremum is taken over every $\varphi \in \mathcal{C}(b,c,\nu)$, every $\rho_1 \in \mathcal{D}(\lambda_1 a, \mu, \gamma_1)$ and $\rho_2 \in \mathcal{D}(\lambda_1 a, \mu, \gamma_2)$ with $\int_{\gamma_1} \rho_1 = \int_{\gamma_2} \rho_2 = 1$, and every pair of local stable leaves γ_1, γ_2.

Let θ_1 and θ_2 be the projective metrics associated to $\mathcal{D}(\gamma_1)$ and $\mathcal{D}(\gamma_2)$, respectively. First, we use condition (B) to get

$$\int_{\gamma_1} \varphi\rho_1 \geq \exp(-b\theta_1(\rho_1, \mathbf{1})) \int_{\gamma_1} \varphi\mathbf{1} \quad \text{and} \quad \int_{\gamma_2} \varphi\rho_2 \leq \exp(b\theta_2(\rho_2, \mathbf{1})) \int_{\gamma_2} \varphi\mathbf{1}$$

where the same symbol $\mathbf{1}$ denotes two slightly different objects, namely, the constant function on γ_1 satisfying $\int_{\gamma_1} \mathbf{1} = 1$ and the constant function on γ_2 with $\int_{\gamma_2} \mathbf{1} = 1$. Let D_1 be some uniform upper bound for the θ-diameter of $\mathcal{D}(\lambda_1 a, \mu, \gamma) \subset \mathcal{D}(\gamma)$, $\gamma \in \mathcal{F}^s_{loc}$; cf. Lemma 3.2. Then

$$\exp(-D_1) \leq \exp(-b\theta_1(\rho_1, \mathbf{1})) \leq 1 \leq \exp(b\theta_2(\rho_2, \mathbf{1})) \leq \exp(D_1).$$

Finally, let $\tilde{\mathbf{1}}\colon \gamma_2 \to \mathbb{R}$ be given by $\tilde{\mathbf{1}}(x) = \mathbf{1}(\pi(x))\,|\det D\pi(x)|$, where $\pi = \pi(\gamma_2, \gamma_1)$. By (p1), both $\mathbf{1}$ and $\tilde{\mathbf{1}}$ belong to $\mathcal{D}(a_0, 1, \gamma_2)$. On the other hand, recall (7),

$$\mathcal{D}(a_0, 1, \gamma_2) \subset \mathcal{D}(a_0, \mu, \gamma_2) \subset \mathcal{D}(a/2, \mu, \gamma_2).$$

Let D_0 be a uniform upper bound for the θ_2-diameter of $\mathcal{D}(a/2, \mu, \gamma_2) \subset \mathcal{D}(\gamma_2)$. Then, using conditions (B) and (C) for the function φ,

$$\frac{\int_{\gamma_2} \varphi\mathbf{1}}{\int_{\gamma_1} \varphi\mathbf{1}} \leq \frac{\int_{\gamma_2} \varphi\mathbf{1}}{\int_{\gamma_2} \varphi\tilde{\mathbf{1}}} \frac{\int_{\gamma_2} \varphi\tilde{\mathbf{1}}}{\int_{\gamma_1} \varphi\mathbf{1}} \leq \exp(b\theta_2(\mathbf{1}, \tilde{\mathbf{1}}))\exp(c\, d(\gamma_1, \gamma_2)^\nu) \leq \exp(b\, D_0 + c).$$

We conclude that

$$\frac{\int_{\gamma_2} \varphi\, \rho_2}{\int_{\gamma_1} \varphi\, \rho_1} \leq \exp(2b\, D_1 + b\, D_0 + c)$$

for all the φ, ρ_1, ρ_2, γ_1, γ_2 under consideration, which completes the proof of (19). Altogether, we have shown that

$$\frac{\int_{\gamma''} (\mathbf{L}\,\varphi_2)\rho''}{\int_{\gamma''} (\mathbf{L}\,\varphi_1)\rho''} \cdot \frac{\int_{\gamma'} (\mathbf{L}\,\varphi_1)\rho'}{\int_{\gamma'} (\mathbf{L}\,\varphi_2)\rho'} = \frac{\sum_{j,l} \left(\int_{\gamma_j} \varphi_2 \bar{\bar{\rho}}_j \cdot \int_{\gamma'_l} \varphi_1 \bar{\rho}_l\right)}{\sum_{j,l} \left(\int_{\gamma''_j} \varphi_1 \bar{\bar{\rho}}_j \cdot \int_{\gamma'_l} \varphi_2 \bar{\rho}_l\right)} \leq \Upsilon_0^2,$$

where

$$\Upsilon_0 = \frac{\Upsilon_2}{\Upsilon_1} \exp(2a + 2b\, D_1 + b\, D_0 + c)$$

for every $\varphi \in \mathcal{C}(b,c,\nu)$, leaves $\gamma', \gamma'' \in \mathcal{F}^s_{loc}$, and normalized densities $\rho' \in \mathcal{D}(\gamma')$ and $\rho'' \in \mathcal{D}(\gamma'')$. Then the Θ_+-diameter of $\mathbf{L}(\mathcal{C}(b,c,\nu))$ is bounded by $\log \Upsilon_0^2$. ∎

In what follows we let $\tau_2 = (1-e^{-D_2}) < 1$. Then, in view of our Proposition 3.6 and proposition 2.3 of [13], the operator $\mathbf{L}$ is a τ_2-contraction for the projective metric Θ.

3.4 An invariant measure for the root map. We will now use the sequence $(\varphi_n = \mathbf{L}^n\, 1)_n$ to construct the measure $\mu_{\mathcal{A}}$ we promised in the beginning of this section. Then, in the next subsection, we will spread this measure to all $\mathcal{M}$, constructing the SRB-measure for f, that we will name μ_0.

It follows, from $\mathbf{L}$ being a contraction, that

$$\Theta_+(\varphi_k, \varphi_l) \le \Theta(\varphi_k, \varphi_l) \to 0 \quad \text{(exponentially fast) as } k, l \to \infty.$$

We shall use the statement of weak*-convergence given by the next proposition to construct the measure $\mu_{\mathcal{A}}$.

For the proof we need the important fact that the local stable foliation is absolutely continuous (see Appendix A of [13]): projections along the leaves of $\mathcal{F}^s_{loc}$ are absolutely continuous maps with Hölder continuous jacobians. As a consequence, we have, for every m-integrable function ψ,

$$\int_{\mathcal{A}} \psi\, dm = \int \left(\int (\psi \mid \gamma) H_\gamma \right) d\tilde{m}(\gamma),$$

where $H\colon \mathcal{A} \to (0, +\infty)$ is such that $\log H$ is (a_0, ν_0)-Hölder continuous, for some $a_0 > 0$ and $0 < \nu_0 \le 1$, and $H_\gamma = (H|\gamma) m_\gamma$ defines a disintegration of m ($\tilde{m}$ is just the quotient measure induced by the Lebesgue measure m in the space of local stable leaves).

Proposition 3.7: *Given any Θ_+-Cauchy sequence $(\varphi_n)_n$ in $\mathcal{C}(b,c,\nu)$, normalized by $\int_{\mathcal{A}} \varphi_n\, dm = 1$ for all $n \ge 1$, and given any continuous function $\psi : \mathcal{A} \to \mathbb{R}$, the sequence $\left(\int \varphi_n\, \psi\, dm\right)_n$ is Cauchy in $\mathbb{R}$.*

Proof: Consider first the case when $\psi > 0$ and $\log \psi$ is $(a/2, \mu)$-Hölder. In particular, $\log \psi$ is $(a/2, \mu)$-Hölder along each local stable leaf γ (with respect to the induced metric). We write

$$\int_{\mathcal{A}} \varphi_n \psi\, dm = \int \left(\int_\gamma \varphi_n \psi dp_\gamma \right) d\tilde{m}(\gamma) = \int \left(\int_\gamma \varphi_n \psi H_\gamma \right) d\tilde{m}(\gamma)$$

where $H_\gamma = H \mid \gamma$. Note that (ψH_γ) is strictly positive and $\log(\psi H_\gamma)$ is (a, μ)-Hölder, as long as we fix $a \ge 2a_0$ and $\mu \le \nu_0$; cf. (7) and (12). Moreover,

$$\int_{\mathcal{A}} \varphi_n\, dm = \int \left(\int_\gamma \varphi_n\, H_\gamma \right) d\tilde{m}(\gamma)$$

and $H_\gamma > 0$ with $\log H_\gamma$ an (a,μ)-Hölder function. Therefore, given $k, l \geq 1$,

$$\frac{\int_\gamma \varphi_k H_\gamma}{\int_\gamma \varphi_l H_\gamma} \geq \alpha_+(\varphi_k,\varphi_l) \quad \text{and} \quad \frac{\int_\gamma \varphi_k \psi H_\gamma}{\int_\gamma \varphi_l \psi H_\gamma} \leq \beta_+(\varphi_k,\varphi_l) \quad \text{for all } \gamma.$$

On the other hand, $\int_\mathcal{A} \varphi_k \, dm = 1 = \int_\mathcal{A} \varphi_l \, dm$ implies that $\int_{\hat\gamma} \varphi_k H_{\hat\gamma} \leq \int_{\hat\gamma} \varphi_l H_{\hat\gamma}$ for some local leaf $\hat\gamma$. Thus,

$$\frac{\int_\gamma \varphi_k \psi H_\gamma}{\int_\gamma \varphi_l \psi H_\gamma} \leq \frac{\beta_+(\varphi_k,\varphi_l)}{\alpha_+(\varphi_k,\varphi_l)} \frac{\int_{\hat\gamma} \varphi_k H_{\hat\gamma}}{\int_{\hat\gamma} \varphi_l H_{\hat\gamma}} \leq e^{\Theta_+(\varphi_k,\varphi_l)} \quad \text{for all } \gamma,$$

implying that

$$\frac{\int_\mathcal{A} \varphi_k \psi \, dm}{\int_\mathcal{A} \varphi_l \psi \, dm} \leq e^{\Theta_+(\varphi_k,\varphi_l)} \quad \text{for all } k,l \geq 1.$$

As a consequence,

$$\begin{aligned} (20) \qquad \left| \int_\mathcal{A} \varphi_k \psi \, dm - \int_\mathcal{A} \varphi_l \psi \, dm \right| &= \left| \int_\mathcal{A} \varphi_l \psi \, dm \right| \left| \frac{\int_\mathcal{A} \varphi_k \psi \, dm}{\int_\mathcal{A} \varphi_l \psi \, dm} - 1 \right| \\ &\leq \sup |\psi| \left(e^{\Theta_+(\varphi_k,\varphi_l)} - 1 \right), \end{aligned}$$

and the proposition is proved in this case.

Now, for an arbitrary μ-Hölder continuous function $\psi\colon \mathcal{A} \to \mathbb{R}$, we write

$$\psi = \psi_B^+ - \psi_B^-, \quad \text{where} \quad \psi_B^\pm = \frac{1}{2}(|\psi| \pm \psi) + B$$

and $B > 0$ is chosen large enough to ensure that $\log \psi_B^\pm$ is $(a/2,\mu)$-Hölder continuous. The previous argument applies to $\psi_B^\pm$ and so, by linearity, the proposition holds for ψ.

Finally, given any continuous function ψ and any $\varepsilon > 0$, we may take $\tilde\psi$ a μ-Hölder function such that $\sup |\psi - \tilde\psi| \leq \varepsilon$. Then, for every $k, l \geq 1$,

$$\left| \int_\mathcal{A} \varphi_k \psi \, dm - \int_\mathcal{A} \varphi_l \psi \, dm \right| \leq \left| \int_\mathcal{A} \varphi_k \tilde\psi \, dm - \int_\mathcal{A} \varphi_l \tilde\psi \, dm \right| + 2\varepsilon;$$

recall that we suppose $\int_\mathcal{A} \varphi_n \, dm = 1$ for all n. By the previous case, the right hand side is bounded by 3ε if k and l are large enough, and so we have proved that $\int_\mathcal{A} \varphi_n \psi \, dm$ is a Cauchy sequence also in this case. ∎

We are now in a position to introduce the SRB-measure $\mu_\mathcal{A}$ of the map F on $\mathcal{A}$. For that we consider $\varphi_n = \mathbf{L}^n\, 1$, for each $n \geq 1$. Then, by Proposition 3.7, $(\varphi_n)_n$ is a Θ-Cauchy sequence, and so it is also a Θ_+-Cauchy sequence. Moreover,

$$\int_\mathcal{A} \varphi_n \, dm = \int_\mathcal{A} (\mathbf{L}^n\, 1)\, 1 \, dm = \int_\mathcal{A} 1\, (U^n 1) \, dm = \int_\mathcal{A} 1 \, dm = 1, \quad \text{for all } n \geq 1.$$

Then we define $\mu_{\mathcal{A}}$ to be the weak*-limit of $(\mathbf{L}^n\,1)\,m = (F^n)_*\,m$:

$$\int \psi\, d\mu_{\mathcal{A}} = \lim \int_{\mathcal{A}} (\mathbf{L}^n\,1)\psi\, dm = \lim \int_{\mathcal{A}} (\psi \circ F^n)\, dm,$$

for each continuous $\psi\colon \mathcal{A} \to \mathbb{R}$. Clearly, $\mu_{\mathcal{A}}$ is invariant under the map F: for any ψ,

$$\int (\psi \circ F)\, d\mu_{\mathcal{A}} = \lim \int_{\mathcal{A}} (\psi \circ F^{n+1})\, dm = \lim \int_{\mathcal{A}} (\psi \circ F^n)\, dm = \int \psi\, d\mu_{\mathcal{A}}.$$

The following lemma, in rough terms, asserts that $\mu_{\mathcal{A}}$ behaves as an absolutely continuous measure (with respect to Lebesgue measure) *if one quotients out local stable leaves.* Let $\mathcal{F}_0$ be the σ-algebra of Borel sets which are the union of local stable leaves: $B \in \mathcal{F}_0$ if and only if B is a Borel subset of $\mathcal{M}$ and, given any local stable leaf γ, either $\gamma \cap B = \emptyset$ or $\gamma \subset B$. Clearly, $\widetilde{m}$ is just the restriction of m to $\mathcal{F}_0$.

LEMMA 3.8: *There is $K > 0$ such that, for every $\psi \in L^1(\mathcal{F}_0)$,*

$$\frac{1}{K} \int_{\mathcal{A}} \psi\, dm \le \int \psi\, d\mu_{\mathcal{A}} \le K \int_{\mathcal{A}} \psi\, dm.$$

Proof: Let γ, $\tilde{\gamma}$ be local stable leaves and $H_\gamma = H|\gamma$ and $H_{\tilde{\gamma}} = H|\tilde{\gamma}$ be as in the proof of Proposition 3.7. In addition, let $\widetilde{H}_\gamma = (H_\gamma \circ \pi)\,|\det D\pi|$, with $\pi = \pi(\tilde{\gamma}, \gamma)$. Recall that $\log H$ is (a_0, ν_0)-Hölder, with (a_0, ν_0) depending only on f. Therefore, up to choosing a, a_1 larger than a_0, and μ, μ_1 smaller than ν_0, as in (7), (12), we have $H_\gamma \in \mathcal{D}_1(\gamma)$ and $H_{\tilde{\gamma}}$, $\widetilde{H}_\gamma \in \mathcal{D}(a/2, \mu, \tilde{\gamma})$. Then properties (B) and (C) give, for each $\varphi_k = \mathbf{L}^k\,1$,

$$\frac{\int_\gamma \varphi_k\, H_\gamma}{\int_{\tilde{\gamma}} \varphi_k\, H_{\tilde{\gamma}}} = \frac{\int_{\tilde{\gamma}} \varphi_k\, \widetilde{H}_\gamma}{\int_{\tilde{\gamma}} \varphi_k\, H_{\tilde{\gamma}}}\,\frac{\int_\gamma \varphi_k\, H_\gamma}{\int_{\tilde{\gamma}} \varphi_k\, \widetilde{H}_\gamma} \le \exp(b\theta_+(\widetilde{H}_\gamma, H_{\tilde{\gamma}}) + c\, d(\gamma, \tilde{\gamma})^\nu) \le \exp(bD_0 + c)$$

where D_0 is a uniform bound for the θ-diameter of $\mathcal{D}(a/2, \mu, \tilde{\gamma}) \subset \mathcal{D}(\tilde{\gamma})$; see Lemma 3.2. For simplicity, we write $K = \exp(c + bD_0)$. Recalling that

$$\int \left(\int_{\tilde{\gamma}} \varphi_k\, H_{\tilde{\gamma}} \right) d\widetilde{m}(\tilde{\gamma}) = \int \varphi_k\, dm = 1$$

we conclude that $\int_\gamma \varphi_k\, H_\gamma \le K$ for every local stable leaf γ. Then

$$\begin{aligned} \int_{\mathcal{A}} \psi(\mathbf{L}^k\,1)\, dm &= \int \psi(\gamma) \left(\int_\gamma (\mathbf{L}^k\,1) H_\gamma \right) d\widetilde{m}(\gamma) \\ &\le K \int \psi(\gamma)\, d\widetilde{m}(\gamma) = K \int_{\mathcal{A}} \psi\, dm; \end{aligned}$$

note that functions $\psi \in L^1(\mathcal{F}_0)$ are constant on each stable leaf γ. Passing to the limit as $k \to \infty$, $\int \psi \, d\mu_{\mathcal{A}} \leq K \int_{\mathcal{A}} \psi \, dm$. The dual inequality $\int \psi \, d\mu_{\mathcal{A}} \geq K^{-1} \int_{\mathcal{A}} \psi \, dm$ may be derived in just the same way, and so the argument is complete. ∎

Remark 3.9: A more efficient way to proceed in order not to suppose the existence of just one good rectangle is the following: we can suppose that in the first negative iteration, the (pre)image of each rectangle crosses all the rectangles. If this is not the case, one can replace f by f^p for some suitable $p > 0$ in the tower construction. It is a well known fact (cf. [14]) that the exponential decay (and also central limit theorem) for f^p (for some $p > 0$) implies the same result for f. Hence, the trees of local stable manifolds in different rectangles will be comparable. That is, there are constants $0 < n_1 < n_2$ such that, given any γ, γ', the proportion of the number of local stable leaves in their preimages in the intersection of any floor of their trees with a fixed rectangle is between n_1 and n_2. So we can compare the integrals in the preimages of γ and γ' in each rectangle, therefore we can compare $\int_\gamma \mathbf{L}\,\varphi\rho$ and $\int_{\gamma'} \mathbf{L}\,\varphi\rho'$.

3.5 An invariant measure for the original system. In this subsection we construct an invariant measure for f, which will result in the SRB measure. Let $\mathcal{A} = \bigcup\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ be as in the end of section 2, that is, $F|_{\mathcal{A}_j} = f^j|_{\mathcal{A}_j}$. Fixed a j, put $\mathcal{A}_{j,l} = f^l(\mathcal{A}_j)$, for $l = 0, \ldots, j-1$. Since F is a first return map, the sets $\mathcal{A}_{j,l}$ are disjoint, for all pairs j, l. Moreover, they exhaust $(\mathrm{mod}\, 0)$ $\mathcal{M}$. So, let us define the following measure over $\mathcal{M}$:

$$\mu_0'(B) = \mu_{\mathcal{A}}(f^{-l}(B)), \quad \forall B \in \mathcal{A}_{j,l}, \quad B \text{ a Borel set.}$$

The measure of other Borel sets (not contained in one single $\mathcal{A}_{j,l}$) is defined in the natural way. It is obviously f-invariant, because, if B is a Borel set in $\mathcal{A}_{j,l}$ with $l > 1$, we have

$$\mu_0'(f^{-1}(B)) = \mu_{\mathcal{A}}(f^{-l+1} f^{-1}(B)) = \mu_{\mathcal{A}}(f^{-l}(B)) = \mu_0'(B),$$

and if B is a Borel set $\in \mathcal{A}$ we have

$$\mu_0'(f^{-1}(B)) = \mu_0'(F^{-1}(B)) = \mu_{\mathcal{A}}(B) = \mu_0'(B).$$

Now, $\mu_0'(f^{-1}(B)) = \mu_0'(B)$ for general Borel sets by σ-additivity.

Now, we are led to the following lemma:

LEMMA 3.10: *μ_0' is an f-invariant finite measure.*

Proof: All we have to do is to prove that $\mu_0'(\mathcal{M}) < +\infty$. However,

$$\mu_0'(\mathcal{M}) = \mu_0'(\sum_{j=1}^{\infty}\sum_{l=0}^{j-1} \mathcal{A}_{j,l}) = \sum_{j=1}^{\infty} j \cdot \mu_{\mathcal{A}}(\mathcal{A}_j).$$

By Lemma 3.8 we know that

$$\mu_{\mathcal{A}}(\mathcal{A}_j) \leq K \cdot m(\mathcal{A}_j)$$

and, by the second section of our paper, we know that $m(\mathcal{A}_j) \searrow 0$ exponentially fast as $j \to \infty$; then $j \cdot m(\mathcal{A}_j) \searrow 0$ exponentially fast as $j \to \infty$.

Therefore,

$$\mu_0'(\mathcal{M}) = \sum_{j=1}^{\infty} j \cdot \mu_{\mathcal{A}}(\mathcal{A}_j) \leq K \cdot \sum_{j=1}^{\infty} j \cdot m(\mathcal{A}_j) < \infty. \qquad \blacksquare$$

Now, we can define μ_0 as the probability we obtain normalizing μ_0':

$$\mu_0 := \mu_0'/\mu_0'(\mathcal{M}).$$

A priori, the fact $\mu_{\mathcal{A}}$ is F-ergodic, alone, does not ensure that μ_0 is *f-ergodic.* However, in the next section, we prove, not only in the context of this section, but for the general setting of the paper, that μ_0 is Kolmogorov and SRB measure for f.

4. Stochastic properties of the original system

4.1 EXISTENCE AND UNIQUENESS OF THE SRB MEASURE. In this section, μ_0 refers both to the measure we constructed in the last section or the SRB measure we obtain as a direct consequence of section 2 (see remark at the end of that section).

We are able to prove:

LEMMA 4.1: *(f, Λ, μ_0) is Kolmogorov. That is, if $\mathcal{B}$ is the σ-algebra of Borel in Λ, there exists a σ-subalgebra $\mathcal{B}_0 \subset \mathcal{B}$ such that:*

(1) $f^{-1}(\mathcal{B}_0) \subset \mathcal{B}_0$.

(2) $\bigcap_{n\geq 0} f^{-n}(\mathcal{B}_0) = \mathbf{1}$, *where $\mathbf{1}$ is the σ-algebra consisting in sets equal to Λ or $\emptyset$, modulo zero-measure sets.*

(3) $\bigvee_{n\geq 0}\{B \in \mathcal{B}; f^{-n}(B) \subset \mathcal{B}_0\} = \mathcal{B}(\mathrm{mod}\, 0)$. *That is, the σ-algebra spanned by the Borel sets which have some preimage (of some iterate of f) in $\mathcal{B}_0$ is, modulo zero-measure sets, equal to $\mathcal{B}$.*

Proof: During this proof, all constants written with the greek letter ϵ are positive.

Let us define $\mathcal{B}_0 = \bigvee_{n\leq 0}\mathcal{R}_n$, where $\mathcal{R}_0 = \mathcal{R}$ is the original Markov partition (intersected with the trimmed tower), and $\mathcal{R}_n = f^n(\mathcal{R}_0), n < 0$. Therefore, $\mathcal{B}_0$ is (modulo zero-measure sets) the σ-algebra spanned by non-positive iterates of rectangles in partition $\mathcal{R}_0$. Since such iterates are Borel sets of Λ, we obtain that $\mathcal{B}_0 \subset \mathcal{B}$.

(1) Given $B_0 \in \mathcal{B}_0$, by lemma 5.2 of [9], B_0 is approximated (in measure) by a disjoint union of elements of the partition $\bigvee_{h=0}^{k}\mathcal{R}_h, k < 0$, if $\|k\|$ is big enough. So, there is no loss of generality in supposing that $B_0 \in \bigvee_{h=0}^{k}\mathcal{R}_h$, for some $k \in -\mathbb{N}$. By the Markovian property, the preimage $f^{-1}(B_0)$ of B_0 is contained in a disjoint union of elements of $\bigvee_{h=0}^{k-1}\mathcal{R}_h$, and so it is in $\mathcal{B}_0$. Therefore, $f^{-1}(\mathcal{B}_0) \subset (\mathcal{B}_0)$.

(2) Let us start by defining the global stable manifold of $x \in \Lambda$. Since we know that local central manifolds in Λ (we are talking about the ones we have after we applied the elimination algorithm) are, in fact, local stable manifolds (in the sense that, if x, y belongs to such a manifold, $d(f^n(x), f^n(y)) \to 0$ and the Liapounov exponents calculated along central directions are negative), we define the global stable manifold Ω_x of $x \in \Lambda$ as being the points $y \in \Lambda$ such that $d(f^n(x), f^n(y)) \to 0$ as $n \to \infty$. (Notice that we conventioned to call Λ the set we obtained after we applied our elimination algorithm.)

Of course, each global stable manifold will be a union of local central manifolds, and each local central manifold will be contained in just one global stable manifold. An obvious fact is that the image by f of a global stable manifold is another global stable manifold.

So, let B be such that $\mu_0(B) > 0$ and, for every $n \geq 0$, there is $B_n \in \mathcal{B}_0$ such that $B = f^{-n}(B_n)$. That is, we have $f^n(B) \in \mathcal{B}_0$, for all $n \geq 0$. In this case, if $x \in B$, then $\Omega_x \subset B$. This is because, if some local central manifold γ is in $B \cap \Omega_x$ and another, say γ', is in $B^c \cap \Omega_x$, then that will be an iterate m such that $f^m(\gamma)$ and $f^m(\gamma')$ will be in the same local central leaf. However, such a leaf must be in $f^m(B) = B_m$, since $B_m \in \mathcal{B}_0$. This is a contradiction with the fact that $\gamma' \subset B^c$, therefore $f^m(\gamma') \subset f^m(B^c) = (f^m(B))^c$.

On the other hand, we are under the assumption that $\mu_0(B) > 0$, and we know that $\mu_0(B) = \mu_0(B_n)$, $\forall n$. Given $\epsilon > 0$, let us take $B_0 \in \bigvee_{h=0}^{k}\mathcal{R}_h$, for some $k < 0$, such that $\mu_0(B_0 \cap B)/\mu_0(B_0) > 1-\epsilon$. So, for some n_1, there is some element B_1 in partition $\mathcal{R}_{n_1}$, such that $B_1 \subset B_0$ and $\mu_0(B \cap B_1)/\mu_0(B_1) > 1-\epsilon$.

Because the Markov partition $\mathcal{R}$ is topologically mixing, there is n_2 such that, for any $R \in \mathcal{R}$, $f^{n_2}(R)$ crosses all rectangles in the Markov partition. Therefore, $U_1 = f^{n_1+n_2}(B_1)$ crosses all rectangles in $\mathcal{R}$.

It follows from the fact that the SRB measure is absolutely continuous that $m_u(\Gamma_1 \cap B)/m_u(\Gamma_1) > 1-\epsilon'$, for all unstable leaf Γ_1 in B_1, $\epsilon' \to 0$, as $\epsilon \to 0$.

Calling $U = f^{n_1+n_2}(B)$, using our distortion bounds in the unstable directions for leaves in U_1 (see Lemma 2.11), we have that

$$m_u(f^{n_1+n_2}(\Gamma_1) \cap U)/m_u(f^{n_1+n_2}(\Gamma_1)) > 1-\epsilon'', \quad \epsilon'' \to 0 \quad \text{as } \epsilon' \to 0.$$

Note that for any $n_1' < 0$, any unstable leaf Γ contained in some element of $\mathcal{R}_{n_1'}$ will have the same bounds for an unstable volume of the images $f^{n_1'+n_2}(\Gamma)$ and distortion in unstable directions. So, we do not have to be worried about the fact that B_1 and n_1 depend on ϵ.

Since U also consists in global stable leaves, and the stable foliation is absolutely continuous (it coincides a.e. with Pesin's stable foliation), we have that $\mu_0(B) = \mu_0(U) > 1-\epsilon'''$, for some $\epsilon''' \to 0$ as $\epsilon'' \to 0$. This implies that $\mu_0(B) = 1$ and finishes this part of the proof.

(3) For this proof we use the following theorem (see [9], chapter 0, theorems 5.4 and 5.5):

THEOREM 4.2: *Let $(\Lambda, \mathcal{B}, \mu_0)$ be a probability space where Λ is a separable metric space, and $\mathcal{B}$ is the Borel σ-algebra of Λ. Let $\mathcal{P}_n$, $n = 0,1,2,\dots$ be a sequence of partitions such that*

$$\lim_{n\to+\infty}(\sup_{P\in\mathcal{P}_n} \operatorname{diam}(P)) = 0.$$

Then, $\bigvee_{n=0}^{\infty} \mathcal{P}_n = \mathcal{B} (\operatorname{mod} 0)$.

We recall that the 0-th floor $\mathcal{A}$ of the tower is divided into subsets $\mathcal{A}_{j,m}, j > 0, m > 0$, such that $F|_{\mathcal{A}_{j,m}} = f^j|_{\mathcal{A}_j}$ and all points in $\mathcal{A}_{j,m}$ belong to the same j-cylinder. By construction, such subsets are elements of $\mathcal{B}_0$. So, we define the (infinite countable) partition $\mathcal{P}$ of Λ given by $\mathcal{P} := \{P_{j,m,l}, 0 \le l < j\}$, where $P_{j,m,l} = f^l(\mathcal{A}_{j,m})$. This implies that given any set $P \in \mathcal{P}$, there exists a negative p such that $f^p(P) \in \mathcal{B}_0$. Given any set $P = P_{j,m,l}$ in $\mathcal{P}$ and a natural n, $f^n(P)$ has central stable diameter diam^{cs} (defined as the supremum of the diameters of the intersections of local center-stable leaves with P) bounded by

$$const \cdot \operatorname{diam}^{cs}(\mathcal{A}) \cdot \varsigma^{l+n},$$

where the constant depends only on some uniform bound for the curvature of the local central leaves.

Now, just take $\mathcal{P}_n = \bigvee_{i=-n}^{n}(f^i(\mathcal{P}))$. The contraction in unstable directions we have if we iterate backwards and the bound we just obtained above (for the center-stable direction) imply that $\lim_{n\to+\infty}(\sup_{P\in\mathcal{P}_n}\operatorname{diam}(P)) = 0$. By Theorem 4.2, $\bigvee_{n\geq 0}\mathcal{P}_n = \mathcal{B}(\operatorname{mod} 0)$. Since the partitions we defined are contained in $\bigvee_{n\geq 0}\{B \in \mathcal{B}; f^{-n}(B) \subset \mathcal{B}_0\}$, this finishes the proof of (3) and our lemma. ∎

So far, we have proved that (f, μ_0) is mixing, since every Kolmogorov system is mixing (and so, ergodic, too). We can now obtain the SRB property (m is the ambient Lebesgue measure):

COROLLARY 4.3: *μ_0 is the unique SRB measure of f in $\mathcal{M}$. The measure μ_0 is ergodic and satisfies*

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x)) \to \int \varphi\, d\mu_0 \quad \textit{as } n\to\infty,$$

for every continuous function $\varphi\colon \mathcal{M}\to\mathbb{R}$ and m-almost all $x\in\mathcal{M}$. In particular, μ_0 is the unique SRB-measure for f in $\mathcal{M}$.

Proof: Given any continuous φ and any pair of points x_1, x_2 belonging to the same stable leaf, then it is easy to see that

$$\frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x_1)) \text{ converges} \quad\Leftrightarrow\quad \frac{1}{n}\sum_{j=0}^{n-1}\varphi(f^j(x_2)) \text{ converges},$$

and in that case the two limits are the same. Let A be the set of points $x\in\mathcal{M}$ such that $\lim n^{-1}\sum_{j=0}^{n-1}\varphi(f^j(x))$ does not exist, for some continuous φ. Then A is a union of stable leaves and the ergodic theorem gives $\mu_0(A)=0$.

If $m(A) > 0$, projecting A along the center-stable manifolds (which are absolutely continuous), we would obtain that the Birkhoff limit above would not exist for some set S with positive m_u measure in some unstable leaf $\Gamma\subset\Lambda$. This implies that $\mu_0(A)>0$, since μ_0 is absolutely continuous. This contradicts the last paragraph. So we get $m(A)=0$.

Now, given any continuous φ, define $\tilde\varphi(x) = \lim n^{-1}\sum_{j=0}^{n-1}\varphi(f^j(x))$. We have just shown that $\tilde\varphi$ is defined almost everywhere, with respect to both measures μ_0 and m. Moreover, by the ergodic theorem, $\tilde\varphi$ is constant in stable leaves (in $\mathcal{M}$) and $\tilde\varphi\circ f = \tilde\varphi$ at μ_0-almost every point. The same argument of the last paragraph applied to the set $\{x \in \mathcal{M} : \tilde{\varphi}(f(x)) = \tilde{\varphi}(x)\}$ gives that $\tilde{\varphi} \circ f = \tilde{\varphi}$ at m-almost every point. Then the fact that μ_0 is mixing implies

$$\left| \int_{\mathcal{M}} (\tilde{\varphi} - \int \tilde{\varphi}\, d\mu_0) \phi\, d\mu_0 \right| = \left| \int_{\mathcal{M}} (\tilde{\varphi} \circ f^n)\, \phi\, d\mu_0 - \int \tilde{\varphi}\, d\mu_0 \int \phi\, d\mu_0 \right| \to 0$$

for every $n \geq 0$ and every continuous function ϕ.

As $\tilde{\varphi}$ is constant in local center-stable leaves, because of the same argument we used in the paragraphs above we obtain that the set $\{x \in \mathcal{M} : \tilde{\varphi} \neq \int \tilde{\varphi}\mu_0\}$ must have null Lebesgue measure.

Therefore,

$$\tilde{\varphi} = \int \tilde{\varphi}\, d\mu_0 = \int \varphi\, d\mu_0,$$

m-almost everywhere and μ_0-almost everywhere. This proves ergodicity, the SRB-property, and uniqueness, simultaneously. ∎

4.2 Decay of correlations and central limit theorem. At this point we can deduce decay of correlations and the central limit theorem from the properties of $\mathcal{A}$ that we obtained above. For this, we can choose between at least two different ways. It is not difficult, for instance, to adapt the techniques in [13] of cones and projective metrics to obtain our theorems. However, since this paper may be quite long, we choose here to use the framework of [14] for this purpose.

The following notions are borrowed from [14]:

Definition 4.4 (Hyperbolic product structure)*:* A set $\Lambda' \subset M$ has **hyperbolic product structure**, if there exists a continuous family of unstable disks $\mathcal{F}^u = \Gamma$ and a continuous family of stable disks $\mathcal{F}^s = \gamma$ such that

(i) $\dim \Gamma + \dim \gamma = \dim M$;
(ii) the Γ-disks are transversal to the γ-disks with the angles between them bounded away from 0;
(iii) each Γ-disk meets each γ-disk in exactly one point; and
(iv) $\Lambda' = (\bigcup \Gamma) \cap (\bigcup \gamma)$

Definition 4.5 (s-subsets and u-subsets)*:* We say that a subset S' of a hyperbolic product structure set S is an **s-subset** if it also has a hyperbolic product structure and its defining families can be chosen to be the same unstable family $\mathcal{F}^u$ of S and a subset $\mathcal{G}^s \subset \mathcal{F}^s$ of the stable family of S. We have an analogous definition for **u-subsets**.

We have the following facts about $\mathcal{A}$:

(P1) $\mathcal{A}$ has *hyperbolic product structure*. Moreover, $m_\Gamma(\mathcal{A}\cap\Gamma)>0$, where m_Γ is the conditional Lebesgue measure over Γ.

In fact, the local unstable and local center-stable leaves form two families of disks as in the definition above. As the local center stable foliation is absolutely continuous, and $m(\mathcal{A})>0$, we have $m_\Gamma(\mathcal{A}\cap\Gamma)>0$.

(P2) There is a countable number of *s-subsets* of $\mathcal{A}$ (union of local stable leaves restricted to $\mathcal{A}$), say $\Lambda_1,\Lambda_2,\ldots$, such that

- on each Γ-disk (contained in an unstable manifold), $m_\Gamma\{\mathcal{A}-\bigcup\Lambda_i)\}=0$;
- for each i, $\exists r_i\in\mathbb{N}^+$ such that $f^{r_i}(\Lambda_i)$ is a **u-subset** of $\mathcal{A}$; we require in fact that, for all $x\in\Lambda_i$, $f^{r_i}(\gamma(x))\subset\gamma(f^{r_i}(x))$ and $f^{r_i}(\Gamma(x))\supset\Gamma(f^{r_i}(x))$;
- for each n, there are at most finitely many i with $r_i=n$;
- $\min\{r_i\}\geq$ some r_0 depending only on f.

Note that if the lower bound r_0 was one, we would just take the collection $\{\Lambda_i\}$ as a reenumeration of the collection $\{\mathcal{A}_j\cap C_{j,m}, C_{j,m}$ is a j-cylinder$\}$. In this case, if Λ_i is contained in some $\mathcal{A}_j$, we just put $r_i=j$.

Let us define the countable partition $\mathcal{Q}$ of Λ consisting in

$$\mathcal{Q}:=\{f^l(\mathcal{A}_j),0\leq l<j\}.$$

Since a finite number of sets $\mathcal{A}_j$ may present a return time $j<r_0$, we divide such ones into subsets $\mathcal{A}'_{j,t}$ consisting in points that stay together in the same component of the partition $\mathcal{Q}$ until they have a return to $\mathcal{A}$ with return time t (by f) greater than or equal to r_0. Then, for such points we put $r_i=t$. So, it suffices to take the collection $\{\Lambda_i\}$ to be a reenumeration of

$$\{\mathcal{A}_j\cap C_i j,k; j\geq r_0, C_{j,k} \text{ is a } j\text{-cylinder}\}\cup$$
$$\{\mathcal{A}'_{j,r_i}\cap C_{r_i,k'}; j<r_0, C_{r_i,k'} \text{ is a } r_i\text{-cylinder}\}.$$

(P3) There is a partition $\mathcal{P}$ of $\bigcup f^n(\mathcal{A})$ to which we can define the **separation time** between two points in $\mathcal{A}$, that is, the time their images stay together in the same rectangle of such partition. Such a *separation time* $s_0(.,.)$ has the following properties:

(i) $s_0(.,.)\geq 0$ and depends only on γ-disks containing the two points;

(ii) the number of "distinguishable" n-orbits starting from $\mathcal{A}$ is finite for each n;

(iii) for x, $y\in\Lambda_i$, $s_0(x,y)\geq r_i+s_0(f^{r_i}(x),f^{r_i}(y))$.

In fact, just take $\mathcal{P}$ as the restriction of $\mathcal{R}$ to Λ. Then, s_0 will just be the time that two points in $\mathcal{A}$ stay together in the same rectangle of $\mathcal{R}$ (or $\mathcal{P}$), and it is, by definition, non-negative. As points in the same γ stay together forever,

s_0 is constant in γ. This confirms property (i). If we fix n, the number of "distinguishable" n-orbits starting from $\mathcal{A}$ is just the number of n-cylinders that intersect $\mathcal{A}$. Property (iii) is a trivial consequence of the definition of the Λ_i.

We saw that there exist $C' > 0$, $0 < \varsigma < 1$, $0 < \lambda_u < 1$ such that the following hold for all $x, y \in \mathcal{A}$:

(P4) We have contraction along γ-disks. For $y \in \gamma(x)$, $d(f^n(x), f^n(y)) \leq C'\varsigma^n, \forall n \geq 0$.

This we have by construction of $\mathcal{A}$ and the tower. By Propositions 2.9 and 2.19, we have $\|Df^n(z)|E^{cs}\| \leq \varsigma^n$, for $z \in \mathcal{A}$ and for all $n > 0$. Therefore, the center-stable manifolds of $\mathcal{A}$ are in fact stable manifolds, with the contraction above.

(P5) Backward contraction and distortion along Γ. For $y \in \Gamma(x)$ and $0 \leq k \leq n < s_0(x,y)$, we have

(a) $d(f^n(x), f^n(y)) \leq C \cdot \lambda_u^{s_0(x,y)-n}$;

(b)

$$\log \prod_{i=k}^{n} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C \cdot \lambda_u^{s_0(x,y)-n}.$$

(a) is due to the fact that x, y belong to the same $s_0(x,y)$-cylinder, and the same unstable leaf. The proof of (b) is similar to the proof of Lemma 3.5 between equations (11) and (13).

(P6) Convergence of $D(f^i|\Gamma)$ and absolute continuity of the local center-stable foliation.

(a) There exists $0 < \alpha < 1$ s.t. for $y \in \gamma(x)$,

$$\log \prod_{i=n}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(y))} \leq C' \cdot \alpha^n, \quad \forall n \geq 0.$$

(b) For $\Gamma, \Gamma' \in \mathcal{F}^u$, if $\Theta\colon \Gamma \cap \mathcal{A} \to \Gamma' \cap \mathcal{A}$ is defined by $\Theta(x) = \gamma(x) \cap \Gamma'$, then Θ is absolutely continuous and

$$\frac{d(\Theta_*^{-1} m_{\Gamma'})}{dm_\Gamma}(x) = \prod_{i=0}^{\infty} \frac{\det Df^u(f^i(x))}{\det Df^u(f^i(\Theta(x)))}.$$

Again, the proof of (P6)(a) looks like the proof of Lemma 3.2. Taking x and y in the same γ, their distance is (uniformly) bounded by the supremum of the diameters of the center-stable leaves, which we will call d_s. As in Lemma 3.2, we have

$$\log \prod_{i=n}^{\infty} \frac{|\det(Df^u)(f^i(x))|}{|\det(Df^u(f^i(y))|} \leq$$

$$\sum_{i=n}^{\infty} |\log|\det(Df^u(f^i(x)))| - \log|\det(Df^u(f^i(y))|| \leq \text{ (by (P4) above)}$$
$$\sum_{i=n}^{\infty} K_u \cdot d(f^i(x), f^i(y)) \leq K_u \cdot \sum_{i=n}^{\infty} d_s \cdot \varsigma^i \leq K_u \cdot d_s \cdot \alpha^n,$$

where K_u is a uniform bound for the Lipschitz constant of Df^u. Property (P6)(b), in our context, is a consequence of (P3)–(P6)(a).

(P7) There exists $C_0' > 0$ and $\theta_0 < 1$ s.t. for some $\Gamma \in \mathcal{F}^u$,

$$m_\Gamma\{x \in \Gamma \cap \mathcal{A} : r_i(x) > n\} < C_0'\theta_0^n, \quad n \geq 0.$$

The proof of this fact is a corollary of the proof of Proposition 2.12. By that proposition, we know that the set of points of $\mathcal{A}$ that stay a long time to have its first return (to $\mathcal{A}$) is an exponentially small (conditional) Lebesgue measure set. Therefore, proceeding as in (P2), we have that the fact above is valid for the subset $\mathcal{S}_1$ of points x in $\mathcal{A}$ such that $r_i(x) > n$ is the *first* return time to $\mathcal{A}$. It is easy to see that the same property is valid for the subset $\mathcal{S}_2$ of points x such that $r_i(x) > n$ is the *second return* time and so on, until, at most, $\mathcal{S}_{r_0}$. Since the number of sets $\mathcal{S}_{-}$ we constructed is finite, we have that their union satisfies (P7), as we wanted to prove.

Finally, we have

(P8) μ_0 is the unique SRB measure of f (in Λ) and (f^n, μ_0) is ergodic $\forall n \geq 1$.

This is valid because we proved that (f, μ_0) is mixing, therefore (f^n, μ_0) is mixing for all n, therefore (f^n, μ_0) is ergodic, for all n. (Indeed, (f^N, μ_0) mixing, for some fixed N, is equivalent to (f, μ_0) mixing.)

Having proved all these facts, our Theorem A can be easily obtained from the following result of L. S. Young:

THEOREM 4.6 (Exponential decay of correlations [14]): *Let f: $M \to M$ be a C^2 diffeomorphism on a compact manifold M, admitting a subset $\mathcal{A} \subset M$ with properties (P1)–(P8). Then (f, μ_0) has exponential decay of correlations in the space of ν-Hölder continuous functions. That is, for $\nu > 0$, if we define*

$$\mathcal{H}_\nu := \{\varphi : M \to \mathbb{R}/\exists C s.t. |\varphi(x) - \varphi(y)| \leq C \cdot d(x,y)^\nu, \forall x, y \in M\},$$

then, for functions φ, $\psi \in \mathcal{H}_\nu$, there exist $K = K(\varphi, \psi)$ and $\tau = \tau(\nu) > 0$ such that

$$\left| \int \varphi(\psi \circ f^n)\, d\mu_0 - \int \psi\, d\mu_0 \int \varphi\, d\mu_0 \right| \leq K \cdot \tau^n.$$

Similarly, Theorem B follows from:

THEOREM 4.7 (Central limit theorem [14]): *Under the same hypotheses of the last theorem, every $\varphi \in \mathcal{H}_\nu$ with $\int \varphi d\mu_0 = 0$ satisfies the central limit theorem with respect to (f, μ_0), with $\sigma = 0$ if and only if $\varphi = \psi \circ f - \psi$, for some $\psi \in L^2(\mu_0)$.*

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